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Survey of Layer Structures and Coordinate Transformations Eliminating Layers

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Abstract

Researchers in the Singular-Perturbation-Problems (SPP) community have largely focused on developing codes to solve problems with exponential layers. Such codes, while generally effective in solving the problems for which they were designed, are difficult or impossible to adapt to problems with non-exponential layers, which are incomparably broader than exponential layers. The purpose of this paper is to demonstrate the need for increased efforts to study non-exponential layers and to develop appropriate new approaches to solving problems involving such layers, in particular new grid-clustering algorithms that rely on the layer structure to solve both model and practical problems. The paper reviews recent achievements in this area. It summarises the current state of knowledge on layer structures, including descriptions of the types of layers discovered so far, with emphasis on non-exponential ones. The paper outlines model problems involving such layers and the conditions under which they occur, describes specific techniques for generating layer-eliminating coordinate transformations and layer-resolving grids, outlines methods for solving certain model problems, and discusses possible applications of the methods to the numerical analysis of practical problems involving layers. It illustrates the theoretical and numerical methodology for a specific problem involving non-exponential layers.

Keywords: Small Parameter, Turning Points, Interior and Boundary Layers, Numerical Grids, Uniform Convergence.

1. INTRODUCTION

Boundary and interior layers present serious obstacles to the efficient calculation of equations modelling many technical problems. It is a difficult task to develop uniformly convergent algorithms for solving such problems. Resources provided by numerical grids can significantly reduce the adverse effects of these layers on the accuracy of numerical experiments. Their efficient application, however, requires a detailed knowledge of the layers themselves: their types and structures; situations

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in which they are encountered; and means for struggling with them, in particular, rules for grid clustering in layers.

Think about methods in vaccine research: biological researchers carefully investigate the features of each of the great variety of viruses which have been discovered; by this method, suitable remedies for many divergent strains of disease have been contrived. To win the fight against layers, experts in numerical methods ought to act similarly, but largely do not. While universally acknowledging the very great importance of layers in both pure and applied mathematical science, researchers largely continue to bear down on the narrow class of problems having mainly exponential layers, using grids suitable only to such layers, making little effort to expand the range of problems to include those having other types of layers.

In the more than 50 years since Bakhvalov published his paper [1], the conversation within the SPP community has centered largely on problems with a small parameter ($\varepsilon = 0$) affecting the higher derivatives, having exponential layers. The well-known and widely accepted numerical grids developed by Bakhvalov and Shishkin [2] are highly efficient for problems having only exponential-type layers, typically represented by functions $\exp(-bx/\varepsilon^k)$. Such layers mainly occur in problems for which the solutions of reduced ($\varepsilon = 0$) problems do not have singularities.

Non-exponential types of layers require different forms of layer-resolving grids — grids above and beyond those of Bakhvalov and Shishkin. Their grids are not suitable for tackling important problems whose solutions have incomparably wider layers: power, logarithmic, and hybrid-type boundary and interior layers (see [3] and [4]), and also require knowledge of the constant b affecting the width of the exponential layer, knowledge of which is not always available, as, for example, for boundary layers in viscous-gas flows [5] or for interior layers in solutions to the quasilinear problems discussed in [3]. For non-exponential types of layers, other forms of layer-resolving grids are needed, which employ layer-eliminating coordinate mappings, in particular, those described in [6] and in [4]. The very popularity of the grids contrived by Bakhvalov and Shishkin seems to hinder researchers from considering other problems with non-exponential layers, which are not handled by these grids. Many researchers concerned with solving practical problems prefer to use their simple, unvaried approaches of grid clustering in zones of layers, without regard to the layers' individual features, with the unfortunate consequence that many papers on SPP are now appearing in the literature which are riddled with errors and false claims, with the further consequence that the area of SPP is becoming heavily polluted with bad papers.

The majority of the SPP community is too blinkered with exponential blinders, as it were, seeming to be unaware even of the existence of other layers, to see the nearly virgin territory presented by problems with non-exponential layers, at the same time claiming boldly that they have developed parameter-uniform numerical methods for the solution of problems having wide application in such fields of the physical sciences and engineering as control theory, electrical networks, lubrication theory, etc. Moreover, some authors, by using defective approaches, are contriving to prove that even well-known problems with interior turning points having power-of-type-2 layers now allegedly have exponential layers, see for example [7] and a response [8] to this paper. Unfortunately, numerous papers on SPP are now surfacing in the literature which are plagued with errors and false claims. As a result, the field of SPP is now heavily contaminated with poor papers. The authors, who are responsible for estimating solution derivatives, must improve their skills in handling problems with turning points that have non-exponential layers if they are to avoid serious mistakes which reflect poorly on everyone involved, including journals publishing the papers as well as the authors themselves.

In contrast to the abundant attention paid to problems having exponential layers, problems having non-exponential layers are in oblivion, like orphans, at present nearly unexplored. The author of the present paper is committed to studying both exponential and non-exponential types of layers (see [3]), developing appropriate new approaches of grid clustering for solving problems having such layers (see [4]), and contributing to the investigation of practical problems (see [9]). The author is certain that these activities hold great interest, and urges the elder generation of the SPP community to direct young researchers to explore regions of the field far more innovative and profitable, instead of clinging doggedly to the exponential theme. This paper aims at awakening the interest — of at least some members of the SPP community — in the multipolar area of problems having non-exponential as well as exponential layers. It is clear that the potential of grid-generation technologies can only be realized with a proper understanding of the multipolar character of layer structures. The author hopes to persuade the SPP community that the current bias in favour of problems with exponential layers represents a crowded collective treading of water which creates a bottleneck in the development of algorithms for solving practical, real-world, problems.

The present paper stresses the necessity of studying the variety of layers, their properties, and methods for handling them while solving practical problems. It surveys current achievements in this direction, which include: descriptions of model problems having layers of various types, and conditions in which they are encountered; specific techniques for investigating them; grid-generation algorithms for solving model problems; and possible approaches to applying the methods to analyze practical problems having layers both discovered and as yet undiscovered. Insights contained may be successfully employed in research to discover new techniques for solving problems with layers. Some of these aspects were not sufficiently covered in previously published surveys [10]–[20].

2. GENERAL MODEL PROBLEM

Typical for the theoretical study of qualitative features arising in solutions to problems having layers along a coordinate x transversal to the layers is the following boundary-value problem for an ordinary differential equation of the second order with a small parameter ε in the diffusion coefficient:

$$\begin{aligned} &-(\varepsilon + d(x))^{\nu} u'' + v(x,u)u' + f(x,u) = 0 , \qquad l_0 < x < l_1 , \\ &u(l_0,\varepsilon) = A_0, \quad u(l_1,\varepsilon) = A_1 , \end{aligned} \tag{1}$$

where $1 \gg \varepsilon > 0$, $d(x) \geq 0$, $d(x) = 0$ at some or all points of the interval $[l_0, l_1]$. This simple problem allows one to gain rather significant understanding of the issues involved in real physical processes, in particular, those modelled by Navier-Stokes equations. Problem (1) is quite amenable to qualitative studies of its solutions' properties. Though representing a highly idealized case, it nevertheless gives solid knowledge about the possible qualitative features of solutions in boundary and interior layers; in particular, it provides a rather profound understanding of the variety and complexity of the singularities arising in practical applications.

The case with a constant diffusion coefficient ($d(x) = 0$) in (1) is widely studied in the literature: [21, 22, 23, 3]. This problem with $d(x) = x$ was formulated in the monograph of Polubarinova–Kochina [24] to model filtration of a liquid in the neighborhood of a circular orifice of a small radius $r = \varepsilon$, while that with $\nu = 2$ appears in the physics of motion of charges viewed as classical particles (see Zamaraev, Khairutdinov, and Zhdanov [25]). Problem (1) with $d(x) = x$, $v(x,u) = a(x)$, $\nu = 1$, $l_0 = 0$ and arbitrary $a(x)$, while for $\nu \geq 2$, but without a boundary turning point, i.e., when $a(0) > 0$, was analysed in [3], Sect. 3.4, while the case of a boundary turning point is discussed in Sect. 4 of this paper. An evolutionary problem related to (1) with $d(x) = x$, $\nu = 2$, modelling diffusion-drift motion of charges, was originally investigated numerically by using special grids in [26]. One more problem with $d(x) = x^2$, $\nu = 1$ and a single interior turning point was considered in paper [27], while its generalization to evolutionary problems was discussed in [28].

2.1 Classification of Layers

Layers are classified by the qualitative behaviour of the corresponding singular functions inside the layers (see [3], pp. 12–23), in particular, by their derivatives. At present, basic layers (exponential, power-of-type-1 and -type-2, and logarithmic) and their combinations (hybrid types of layers) are known for solutions to problem (1). Of course, solutions to this problem, not to mention to Navier-Stokes equations, may have new types of layers that have not yet been discovered.

2.1.1 Exponential layers

The lion's share of interest in the SPP community is paid to exponential types of layers, i.e., when solution derivatives are estimated in the vicinity of a point x_0 as

$$\mid u^{(i)}(x,\varepsilon) \mid \leqslant M[\varepsilon^{-ik}\exp(-b|x-x_0|/\varepsilon^k)+1] \ , \quad b>0 \ , \quad 1 \leqslant i \leqslant n \ , \quad 0 \leq |x-x_0| \leq m \ , \tag{2}$$

where the constant k here and hereafter is referred to as the scale of the layer. Also, in this equation and hereafter, by m, M, m_i, M_j we designate positive constants independent of ε.

An obvious simple representative of the function, designated as $u_e(x,\varepsilon)$, having the exponential layer near $x = 0$ and satisfying $u_e(0,\varepsilon) = 0$, $u_e'(0,\varepsilon) > 0$ is

$$u_e(x,\varepsilon) = 1 - \exp(-bx/\varepsilon^k) \ , \quad b > 0 \ , \quad x \geq 0 \ . \tag{3}$$

Functions $\mathrm{erf}(x/\varepsilon)$ and $\tanh(x/\varepsilon)$ give other examples of exponential layers. It is well known that such layers appear in solutions to problem (1) with $d(x) = 0$, $\alpha = 1$, namely,

$$-\varepsilon u'' + v(x,u)u' + f(x,u) = 0, \ 0 < x < 1, \ u(0) = A_0, \ u(1) = A_1, \tag{4}$$

near $x = 0$ of scale $k = 1$ when $v(x,u) < 0$ and $f_u(x,u) \geq 0$; and of scale $k = 1/2$ when $f_u(x,u) > 0$ and $v(x,u) = a(x)$, $a(0) = a'(0) = 0$, in particular, if $v(x,u) \equiv 0$, see [3], pp. 92-93 and [29] for a linear case. Less familiar cases of problems having exponential layers are outlined below.

Solutions to problem (1) with $d(x) = x$, $v(x,u) = a(x)$, $f_u(x,u) > 0$, i.e.,

$$-(\varepsilon + x)^{\nu} u'' + a(x)u' + f(x,u) = 0, \ 0 < x < 1, \ u(0) = A_0, \ u(1) = A_1, \tag{5}$$

have exponential boundary layers of scale $k = \nu$ near $x = 0$ when $a(0) > 0,\ \nu \geq 2$, and $f_u(x,u) > 0$ (see [3], Sect. 3.4.2 and [30]). Interior exponential layers have solutions to semi-linear problems

$$-\varepsilon u'' + a(x)u' + f(x,u) = 0,\ 0 < x < 1,\ u(0) = A_0,\ u(1) = A_1, \tag{6}$$

for which a coefficient before the first derivative is a function of an exponential-layer type as in [31] and [32] or coefficients of the equation are discontinuous at interior points as in [33]. Boundary exponential layers have solutions to problem (6) with a boundary multiple turning point (see [3, 29]). Boundary and interior exponential layers may have solutions to quasi-linear problem (4). One such specific case, namely,

$$-\varepsilon u'' + uu' + f(x,u) = 0,\ f(x,0) = 0,\ f_u(x,u) \geq c > 0,\ 0 < x < 1,\ u(0) = A_0,\ u(1) = A_1, \tag{7}$$

was considered in [3], Sect. 4.3, where it was shown that estimates for the first derivative of a solution to (7) are largely described through the values of the boundary conditions A_0 and A_1 and the functions $\varphi_j(x),\ j = 0, 1$, which are solutions to the corresponding initial-value problems obtained from the reduced ($\varepsilon = 0$) problem:

$$\varphi_j' + f(x,\varphi_j)/\varphi_j = 0\ , \quad j = 0,1\ , \quad x > 0 \ \text{ if } j = 0, \quad x < 1 \ \text{ if } j = 1, \quad \varphi_i(i) = A_i\ , \quad i = 0,1\ .$$

The functions $\varphi_j(x),\ j = 0, 1$, are strictly decreasing ($\varphi_j'(x) \leq -c,\ j = 0, 1$). In [3], pp. 128-129, the following was proven

Theorem 1 *Let $u(x,\varepsilon)$ be a solution to problem (7) then $|u'(x,\varepsilon)| \leq M$, $0 \leq x \leq 1$, when either 1. $\varphi_0(1) = A_1$, or 2. $A_0 \geq 0$, $A_1 \leq 0$, and $\varphi_0(1) < A_1$;*

$$|u'(x,\varepsilon)| \leq M\Big[1 + \varepsilon^{-1}\exp\Big(-\frac{m|x - x_0|}{\varepsilon}\Big)\Big]\ , \qquad 0 \leq x \leq 1\ , \tag{8}$$

for some $m > 0$, where $x_0 = 0$ if 1. $A_0 \leq 0$ and $A_1 \leq 0$ or 2. $A_0 > 0$, $A_1 < 0$, and $A_0 + \varphi_1(0) < 0$; $x_0 = 1$ if 1. $A_0 \geq 0$ and $A_1 \geq 0$ or 2. $A_0 > 0$, $A_1 < 0$, and $A_1 + \varphi_0(1) > 0$; x_0 is a root of the equation $\varphi_0(x, A_0) + \varphi_1(x, A_1) = 0$, if $A_0 > 0$ $\ A_1 < 0$, $\varphi_0(1) > A_1$, $A_0 + \varphi_1(0) \geq 0$, and $A_1 + \varphi_0(1) \leq 0$;

$$|u'(x,\varepsilon)| \leq M\Big[1 + \varepsilon^{-1}\exp\Big(-\frac{mx}{\varepsilon}\Big) + \varepsilon^{-1}\exp\Big(\frac{m(x-1)}{\varepsilon}\Big)\Big]\ ,\ 0 \leq x \leq 1\ , \tag{9}$$

for some $m > 0$, if $A_0 < 0$ and $A_1 > 0$.

Estimates for higher derivatives have not yet been discovered.

One more case for an autonomous problem:

$$-\varepsilon u'' + a(u)u' = 0,\ 0 < x < 1,\ u(0) = A_0,\ u(1) = A_1, \tag{10}$$

was formulated in [3], Sect. 1.3.3, as a simple model suitable for analysing possible boundary and interior layers in gas flows. This problem comes down from a system of ordinary differential equations simulating the shock wave structure of a steady heat-conducting gas flow:

$$\begin{aligned}
&c\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\mathrm{d}p}{\mathrm{d}x} - \varepsilon\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = 0\ , && 0 < x < 1\ ,\\
&c\frac{\mathrm{d}e}{\mathrm{d}x} + p\frac{\mathrm{d}u}{\mathrm{d}x} - \varepsilon\Big(\frac{\mathrm{d}u}{\mathrm{d}x}\Big)^2 - \frac{\mathrm{d}}{\mathrm{d}x}\Big(\chi\frac{\mathrm{d}T}{\mathrm{d}x}\Big) = 0\ , &&\\
&c = \rho u\ , \quad (\rho\ u\ e)(0,\varepsilon) = (\rho_0\ u_0\ e_0)\ , \quad (\rho\ u\ e)(1,\varepsilon) = (\rho_1\ u_1\ e_1)\ , &&
\end{aligned} \tag{11}$$

where ρ is the density, u the velocity, p the pressure, T the temperature, e the energy, ε the coefficient of gas viscosity, and χ the coefficient of thermal conductivity. In the case $e = c_v T$, $p = (\nu - 1)\rho e$ we get from system (11)

$$-\varepsilon u'' + c[u + (\nu - 1)e/u]' = 0 , \qquad 0 < x < 1 ,$$
$$-(\varepsilon_1 e')' + c\Big(e - \frac{u^2}{2} + \frac{c_2}{c}u\Big)' = 0 , \qquad 0 < x < 1 ,$$
$$u(0) = u_0 , \quad u(1) = u_1 , \qquad e(0) = e_0 , \quad e(1) = e_1 , \tag{12}$$

where $c = \rho_0 u_0$, $\varepsilon_1 = \chi/c_v$, $c_2 = (-\varepsilon u' + c[u + (\nu - 1)e/u])|_{x=0}$. The functions $u(x)$ and $e(x)$ are monotonic in the layers of their rapid variation. Hence the dependent variables u and e are connected by the relations $e = e(u)$, $u = u(e)$. Therefore, problem (11) can be specified into two coupled problems. In particular, with respect to the dependent variable u, we obtain a two-point boundary-value problem of a simple autonomous quasilinear form (10), where $a(u) = c\frac{\mathrm{d}}{\mathrm{d}u}[u+(\nu-1)e(u)/u]$. An analogous problem can be formulated for the dependent variable e in (11) if ε_1 is a constant.

Solutions to problem (10) can have exponential, power, and hybrid layers. It was shown that the estimates for derivatives of the solution to this problem are largely described through the values of a function $b(u) = \int_{A_0}^{u} a(\eta)\mathrm{d}\eta$ and boundary conditions A_0 and A_1. In particular, solutions to this problem have boundary exponential layers when u_0 is a unique point of the absolute minimum of $b(u)$, and either $u_0 = A_0$ or $u_0 = A_1$ and $a(u_0) \neq 0$, while in the case $b(A_0) = b(A_1)$, $b(u) > b(A_0)$, $A_0 < u < A_1$ and $a(A_j) \neq 0$, $j = 0, 1$, the solutions have an interior exponential layer in the vicinity of a point $x_0 = a(A_1)/(a(A_1) - a(A_0))$. Estimates of solution derivatives for ordinary quasilinear problems (7) and (10) were published in [30], [3], and [34].

Detailed descriptions of numerical and theoretical analysis of problems having exponential layers are presented in surveys [15] and [19]. We describe further the cases of numerical analysis not included in these and other surveys.

Numerical experiments for problem (7) having interior exponential layers were performed by Petrenko in [34] (pp. 235-241) for $f(x, u) = u + u^3$, Vaseva in [6] (Sect. 8.4.5) for $f(x, u) = \exp(x)u$, Karasuljic' in [4] (pp. 131-136) for $f(x, u) = (\cos(\pi x) + 4)u$ and $f(x, u) = (\cos(\pi x) + 2)u$, and by O'Riordan and Quinn in [35] for $f(x, u) = b(x)u$.

Numerical experiments for problem (10) with exponential boundary and interior layers were performed by Vaseva in [6] (pp. 254-261) and by Karasuljic' in [4] (Sect. 4.3.8).

Some analysis for localising discontinuities of the limit solution to problem (10) and (4) with $a(x, u) = z(u)$, $f_u(x, u) \geq c > 0$, was done by Lorenz in [36] and [37]. The same analysis of the limit solution to problem (4) with $a(x, u) = f(x, u) = u$, was done by [38], [39]. The applicability of differential inequalities to this boundary-value problem analysis was discussed in [40].

2.1.2 Power-of-type-1 layers

Derivatives of solutions having power-of-type-1 layers of scale k in the vicinity of a point x_0 are estimated as

$$|u^{(i)}(x,\varepsilon)| \leqslant M[\varepsilon^{kb}/(\varepsilon^k + |x - x_0|)^{b+i} + 1]\ , \quad b > 0\ , \quad 1 \leqslant i \leqslant n\ , \quad 0 \leqslant |x - x_0| \leqslant m\ , \tag{13}$$

and not estimated by (2), since for an arbitrary $b_2 > 0$

$$\varepsilon^{-ik}\exp(-b|x - x_0|/\varepsilon^k) + 1 \le M[\varepsilon^{kb_2}/(\varepsilon^k + |x - x_0|)^{b_2+i} + 1]\ , \tag{14}$$

for some $M > 0$, where M is independent of ε but dependent on b and b_2. An obvious simple representative of the function , designated as $u_{p1}(x,\varepsilon)$, having the power-of-type-1 layer near $x = 0$ and satisfying $u_{p1}(0,\varepsilon) = 0$, $u'_{p1}(0,\varepsilon) > 0$, is

$$u_{p1}(x,\varepsilon) = 1 - \varepsilon^{kb}/(\varepsilon^k + x)^b\ , \quad b > 0\ , \quad x \ge 0\ . \tag{15}$$

Boundary power-of-type-1 layers of scale $k = 1/2$ have solutions to problem (6) with a boundary turning point $x_0 = 0$ or $x_0 = 1$ when $a(x_0) = 0$, $a'(x_0) > 0$ (see [3], Sect. 3.3). Note that problem (6) with a boundary turning point is not always solved by the method of asymptotic expansions, since the reduced problem ($\varepsilon = 0$) may be ill-posed, and thus the values of solutions and/or the derivatives of the reduced problem may be unbounded, though it is independent of ε. Boundary power-of-type-1 layers of scale $k = 1$ have solutions to problem (5) with $\nu = 1$, $-a(0) > 1$ (see [3], Sect. 3.4).

Interior power-of-type-1 layers of scale $k = 1$ have solutions to problem (10) (see [3], Sect. 4.2.3). It was shown that when $b(A_0) = b(A_1)$, $b(u) > b(A_0)$ for $A_0 < u < A_1$ and, besides this, $b'(A_0) = b'(A_1) = 0$, $b''(A_0) \neq 0$, $b''(A_1) \neq 0$, then

$$|u^{(k)}(x,\varepsilon)| \le M[1 + \varepsilon/(\varepsilon + |x - x_0|)^{1+k}]\ , \qquad 0 \le x \le 1\ , \tag{16}$$

where

$$x_0 = \frac{1}{\sqrt{c_0 + c_1}}\Big[\sqrt{c_1} - \varepsilon \ln \varepsilon^{-1}\Big(\frac{d_0}{(c_0)^{3/2}} + \frac{d_1}{(c_1)^{3/2}}\Big)\Big]\ ,$$

$c_i = a'(A_i)/2$, $d_i = a''(A_i)/6$, $i = 1, 2$. It was also shown (see [3], Sect. 4.2.2) that solutions to problem (10) have boundary power-of-type-1 layers of scale $k = 1$ when u_0 is a unique point of the absolute minimum of $b(u)$, and either $u_0 = A_0$ or $u_0 = A_1$ and $a^{(p)}(u_0) \neq 0$, $a^{(i)}(u_0) = 0$, $i < p$ for some $p > 0$.

Numerical experiments with problem (6) having boundary power-of-type-1 layers were carried out by Vaseva in [6], Sect. 8.1.4, Paasonen in [4], Sect. 3.4.3, and by Karasuljic' in [4], Sect. 4.4.1.–4.4.3. Problem (10) having boundary power-of-type-1 layers was analyzed numerically by Vaseva in [6] (pp. 254-256 for $a(u) = u - 1$, $A_0 = 0$, $A_1 = 1$) and by Karasuljic' in [4] (Sect. 4.4.5. for $a(u) = u - 2$, $A_0 = 0$, $A_1 = 2$). Numerical experiments with the same problem (10) but having interior power-of-type-1 layers were conducted by Karasuljic' in [41].

2.1.3 Power-of-type-2 layers

Solutions having power-of-type-2 layers of scale k in the vicinity of a point x_0 are estimated as

$$|u^{(i)}(x,\varepsilon)| \leqslant M[(\varepsilon^k + |x-x_0|)^{b-i} + 1]\ , \quad b > 0\ , \quad 1 \leqslant i \leqslant n\ , \quad 0 \leqslant |x - x_0| \leqslant m\ . \tag{17}$$

An obvious simple representative of the function, designated as $u_{p2}(x,\varepsilon)$, having the power-of-type-2 layer near $x = 0$ and satisfying $u_{p2}(0,\varepsilon) = 0$, $u'_{p2}(0,\varepsilon) > 0$, is

$$u_{p2}(x,\varepsilon) = (\varepsilon^k + x)^b - \varepsilon^{kb}\ , \quad b > 0\ , \quad x \geq 0\ . \tag{18}$$

Boundary power-of-type-2 layers of scale $k = 1$ have solutions to problem (5) with $\nu = 1$, $1 > -a(0) > 0$ (see [3], Sect. 3.4).

Interior power-of-type-2 layers of scale $k = 1/2$ have solutions to problem (4) with $v(x,u) = a(x)$, $a(x_0) = 0$, $a'(x_0) < 0$, $0 < x_0 < 1$ (see [3], Sect. 3.3, or in the linear case, [42]) and solutions to problem (1) with $\nu = 1$, $d(x) = x^2$, $v(x,u) = a(x)$, $f(x,u) = c(x)u - g(x)$, i.e.,

$$-(\varepsilon + x^2)u'' + a(x)u' + c(x)u = g(x),\ -1 < x < 1,\ u(-1) = A_0,\ u(1) = A_1, \tag{19}$$

when $a(0) = 0$, $a'(0) + c(0) > 0$ (see [27]).

Numerical experiments with problem (5) having boundary power-of-type-2 layers were conducted by Vaseva in [6], Sect. 8.2.4, and Paasonen in [4], p.51. Numerical experiments with problem (6) having interior power-of-type-2 layers were carried out by Vaseva in [6], Sect. 8.1.6, Paasonen in [4], Sect. 3.4.3, and by Karasuljic' in [4], Sect. 4.4.6.

2.1.4 Logarithmic layers

Solutions having logarithmic layers of scale k in the vicinity of a point x_0 are estimated as

$$|u^{(i)}(x,\varepsilon)| \leqslant M[(\varepsilon^k + |x-x_0|)^{-i}/\ln(1/\varepsilon) + 1]\ , \quad 1 \leqslant i \leqslant n\ , \quad 0 \leqslant |x - x_0| \leqslant m\ . \tag{20}$$

An obvious simple representative of a function, designated as $u_l(x,\varepsilon)$, having a logarithmic layer near $x = 0$ and satisfying $u_l(0,\varepsilon) = 0$, $u'_l(0,\varepsilon) > 0$, is

$$u_l(x,\varepsilon) = 1 - \ln(\varepsilon^k + x)/\ln(\varepsilon)\ , \quad x \geq 0\ . \tag{21}$$

Boundary logarithmic layers of scale $k = 1$ have solutions to problem (5) with $\nu = 1$, $a(0) = -1$ (see [3], Sect. 3.4). Numerical experiments over this problem with a boundary logarithmic layer were conducted by Vaseva in [6], Sect.8.2.5, Paasonen in [4], Sect. 3.5.2, and by Karasuljic' in [4], Sect. 4.5.

2.1.5 Hybrid layers

Hybrid layers have solutions whose derivatives are estimated by combinations of estimates for single basic layers from (2), (13), (17), and (20), for example, by a combination of power-of-type-2 layer of scale k (13) and power-of-type 1 layers of scale k (17):

$$|u^{(i)}(x,\varepsilon)| \leqslant M[\varepsilon^{kb}/(\varepsilon^k+x)^{b+i}+(\varepsilon^k+x)^{d-i}+1]\ , \quad b>0\ , \quad d>0\ , \\ 1\leqslant i\leqslant n\ , \quad 0\leqslant x\leqslant m\ . \tag{22}$$

Estimates of derivatives of a solution $u(x,\varepsilon)$ having this hybrid layer in the vicinity of an arbitrary, in particular, interior, point x_0 are obtained from formula (22) by substituting $|x-x_0|$ for x.

Estimates for interior hybrid layers in the vicinity of an interior point x_0 can also be represented by the following combination of basic estimates, for example, (13) and (17) with x replaced by $|x-x_0|$:

$$|u^{(i)}(x,\varepsilon)| \leq \begin{cases} M[\varepsilon^{kb}/(\varepsilon^k+x_0-x)^{b+i}+1]\ , & m_1\leq x\leq x_0\ , \\ M[(\varepsilon^k+x-x_0)^{b-i}+1]\ , & m_2\geq x\geq x_0\ . \end{cases} \tag{23}$$

Such a layer is formed by two contiguous layers of power-of-1 and power-of-2 types, both of scale k. Of course, hybrid layers in (22) and (23) may be composed of layers of different types and scales. Such hybrid layers may have solutions to systems of equations (see [3], Chap. 5).

Hybrid boundary layers of scale $k=1/2$ near $x=0$ have solutions to problem (6) with $a(0)=0$, $a'(0)<0$ (see [3], pp. 80-86). These layers are combinations of power-of-type-1 and power-of-type-2 layers, while in the case $f(x,u)=c(x)u+g(x)$ or $f(x,u)=xw(x,u)$, $w_u(x,u)>0$, they are combinations of exponential and power-of-type-2 layers see ([43] and [3], Sect. 3.3.2). It was demonstrated that solutions to problem (19) with $a(0)=0$, $a'(0)+c(0)\leq 0$ exhibit hybrid interior layers of scale $k=1/2$ which are combinations of power-of-first-type and power-of-second-type layers. Interior hybrid layers of scale $k=1$ have also solutions to problem (10) when the function $b(u)=\int_{A_0}^{u} a(\eta)\mathrm{d}\eta$ has two or more points of its absolute minimum (see [3], pp. 125-127).

Numerical experiments with problem (6) having boundary hybrid layers were performed by Vaseva in [6], section 8.1.7 and Paasonen in [4], Sect. 3.4.5. Numerical experiments on problem (10) having interior hybrid layers were carried out by Karasuljic' in [4], Sect. 4.6.1.

Note that such classification of layers presented in this Sect. 2 is extended to cases where i_{th} derivatives for any layer are uniformly bounded for $i<j$, $j>1$, while they are estimated by a corresponding to this layer formula from (2), (13), (17), (20), (22), and (23), where the index $i+j$ replaces the index i in its left part.

2.1.6 Layer zone

A layer zone with respect to the nth derivative of a function $u(x,\varepsilon)$ is the narrowest interval in which this derivative tends to ∞ inside the interval when $\varepsilon\to 0$, while outside it is bounded by a constant M. We denote by $\Delta_e^{b,n}$, $\Delta_{p1}^{b,n}$, $\Delta_{p2}^{b,n}$, and Δ_l^n the width of the corresponding exponential, power-of-type-1, power-of-type-2 and logarithmic layer. Using the basic majorants (2), (13), (17), (20), of the function derivatives in these layers of scale k, we obtain

$$\begin{aligned}
&\Delta_e^{b,n} = \frac{\varepsilon^{kn}}{b} \ln \varepsilon^{-k} , \qquad \Delta_{p1}^{b,n} = m_2 \varepsilon^{kb/(b+1)} , \\
&\Delta_{p2}^{b,n} = 0 , \ \text{ if } \ n \le b , \quad \Delta_{p2}^{b,n} = m_3 , \ \text{ if } \ n > b , \\
&\Delta_l^n = \frac{m_4}{(\ln(1+\varepsilon^{-k}))^{1/n}} .
\end{aligned} \tag{24}$$

Note that the constants k, b, and m_i are individual for each equation in (24). For the interior hybrid layer near a point $x = x_0$ determined by estimates (23) its width related to the nth derivative is $\Delta_{p1}^{b,n} + \Delta_{p2}^{b,n}$, while for the hybrid layer near a point $x = x_0$ determined by estimates (22) its width is $\max\{\Delta_{p1}^{b,n}, \Delta_{p2}^{b,n}\}$.

We see that the widths of the basic types of layers for small values of the parameter ε and the same scale k are related as follows:

$$\begin{aligned}
&\Delta_e^{b,n_1} < \Delta_e^{b,n_2}, \ \Delta_{p1}^{b,n_1} < \Delta_{p1}^{b,n_2}, \ \Delta_l^{n_1} < \Delta_l^{n_2}, \ \Delta_{p2}^{b,n_1} \le \Delta_{p2}^{b,n_2}, \ \text{if } n_1 < n_2; \\
&\Delta_e^{b_1,n} < \Delta_e^{b_2,n}, \ \Delta_{p1}^{b_1,n} < \Delta_{p1}^{b_2,n}, \ \Delta_{p2}^{b_1,n} \le \Delta_{p2}^{b_2,n}, \ \text{if } b_1 > b_2; \\
&\Delta_e^{b,n} \ll \Delta_{p1}^{b,n} \ll \Delta_l^n, \text{for } n > 0; \ \Delta_l^n \ll \Delta_{p2}^{b,n}, \ \ \text{for } n > b.
\end{aligned} \tag{25}$$

The first line of these inequalities indicates that the higher the order of the numerical algorithm and hence the higher the derivatives of the solution in the truncation error, the wider the layer and hence the wider the zone for grid clustering should be. The second line shows that the width of the layer increases with decreasing b. From the third line we can conclude that exponential layers are the narrowest of the layers described. As a result, layer-resolving grids designed for solving problems having exponential layers may not be efficient for solving problems having wider non-exponential layers.

The centre of a layer is a zone where the derivatives have large values compared to the values at other points in the layer. The width of the centre of a layer of scale k is $m\varepsilon^k$. The rest of the layer is called its transition part. The transition part is incomparably wider than the centre of the layer. The value of the respective derivative in the layer is reduced from a large value in the centre, and gradually in the transition part, to a uniformly limited value at the boundary of the layer.

2.1.7 A remark about another type of layers

For a problem which is not of form (1), namely,

$$\varepsilon u'' = (u')^2 , \qquad 0 < x < 1 , \qquad u(0,\varepsilon) = 1 , \qquad u(1,\varepsilon) = 0 ,$$

presented in the monograph by Chang and Howes [45], the exact solution is

$$u(x,\varepsilon) = -\varepsilon \ln[x + (1-x)\exp(-1/\varepsilon)] , \qquad 0 \le x \le 1 ,$$

and consequently

$$u^{(k)}(x,\varepsilon) = (-1)^k (k-1)! \frac{-\varepsilon[1-\exp(-1/\varepsilon)]^k}{[x+(1-x)\exp(-1/\varepsilon)]^k} , \ \ k \ge 1 , \ \ 0 \le x \le 1 .$$

This gives us $|u^{(i)}(0,\varepsilon)| \approx \varepsilon \exp(i/\varepsilon)$ and so this boundary–layer function is not of the types described in sections 2.1.1–2.1.6. One reason for the difficulty is that a solution should be found by the power series in $\exp(-\varepsilon^{-1})/\varepsilon$ rather than ε, which is not comparable.

3. TRANSFORMATIONS ELIMINATING LAYERS

The numerical algorithm advocated by the author of this paper for solving (1) is based on piece-wise smooth layer–damping coordinate transformations $x(\xi,\varepsilon) : [0,1] \to [l_0, l_1]$ in compliance with a basic principle: they are to eliminate singularities of high order of solutions $u(x,\varepsilon)$ at each interval $[a_i, b_i]$ of smoothness; i. e., the high–order derivatives of any concrete solution with respect to the new coordinate ξ are to have the following bounds:

$$\Big|\frac{\mathrm{d}^{\mathrm{i}}}{\mathrm{d}\xi^{\mathrm{i}}} u[x(\xi,\varepsilon),\varepsilon]\Big| \le M\ , \quad i \le n\ , \quad a_i \le \xi \le b_i\ , \tag{26}$$

where the constant M is independent of the parameter ε, and the number n is dependent on the order of the approximation of the problem: the higher the order, the larger the number n will be. With the help of such transformations, any problem can be solved by using high order approximations in the physical interval x on layer–resolving grids defined by mapping the nodes of a uniform grid with suitable coordinate transformations $x(\xi,\varepsilon)$, as in [47]. It is proposed that by using the layer–resolving grids obtained by transformations $x(\xi,\varepsilon)$ satisfying (26), $\varepsilon-$uniform high-order convergence will be demonstrated for schemes of high order in the physical interval x. Moreover, the numerical solution can be interpolated $\varepsilon-$uniformly from grid points to the entire interval with the same high-order accuracy.

Univariate transformations $x(\xi,\varepsilon)$ eliminating singularities of solutions to singularly perturbed problems on the interval $[l_0, l_1]$ can be generated eithe explicitly using functions inverse to basic representatives (3), (15), (18), and (21) of layers or implicitly by solving the following problem:

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\Big[\frac{\mathrm{dx}}{\mathrm{d}\xi}\psi(x,\varepsilon)\Big] = 0\ , \quad x(0,\varepsilon) = l_0\ , \quad x(1,\varepsilon) = l_1\ , \quad 0 \le \xi \le 1\ , \tag{27}$$

where $\psi(x,\varepsilon)$ is a suitable weight function. Such an approach, based on (27) and referred to as the equidistribution principle, has been reported by a number of researchers. The original one-dimensional integral formulation of the equidistribution principle was proposed by Boor in [51], while one-dimensional and multidimensional differential and variational versions of this principle were presented in [52], [53], [54], [55], [62], [63], [64], and [9].

3.1 Bound on the First Derivative

A necessary condition for having such transformations satisfying (26) gives formula (2.26) from [3] for the first derivative of a solution to (1) with $f_u(x,u) > 0$, namely, $\int_{l_0}^{l_1} |u'(x,\varepsilon)|\mathrm{d}x \le M$, i.e., the variation of the solution $u(x,\varepsilon)$ on the interval $[l_0, l_1]$ is uniformly bounded. In this case, a transformation $x(\xi,\varepsilon)$ satisfying (26) exist at least for $n = 1$ (see [3], pp.185-186). Therefore, for the purpose of defining explicitly a layer-damping transformations $x(\xi,\varepsilon) : [0,1] \to [l_0, l_1]$, as in

[4], pp. 44-46, 52-50, applied for generating layer-resolving grids by the formula $x_i = x(i/N, \varepsilon)$, $i = 0, 1, \dots, N$, we must have estimate

$$|u'(x,\varepsilon)| \le \phi(x,\varepsilon) \quad \text{and} \quad \int_{l_0}^{l_1} \phi(x,\varepsilon)\mathrm{d}x \le M \ . \tag{28}$$

Such estimate is valid for the first derivative of functions with exponential, power-of-type-1, power-of-type-2, logarithmic, and hybrid layers described in Sect. 2.

3.2 Bounds on the Higher Derivatives

In order to eliminate singularities of order n higher than 1, one needs a knowledge of certain estimates of solution derivatives up to some order n_1. In particular, the transformation $x(\xi,\varepsilon)$, which eliminates the singularities of $u(x,\varepsilon)$ up to the second order, should, following (26) and [55], be such that $|\mathrm{d}^2 u_1/\mathrm{d}\xi^2| \le M$, i.e.,

$$\Big|\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}\Big(\frac{\mathrm{d}x}{\mathrm{d}\xi}\Big)^2 + \frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}^2 x}{\mathrm{d}\xi^2}\Big| \le M \ , \qquad 0 \le \xi \le 1 \ . \tag{29}$$

It seems that in order to generate the necessary transformation $x(\xi,\varepsilon)$ explicitly, an appropriate estimate is needed not only for the first derivative but for the second derivative of $u(x,\varepsilon)$ as well. It is typically suggested, in applications to singularly perturbed equations, that the required transformation $x(\xi,\varepsilon)$ can be obtained from (27), where $\psi(x,\varepsilon)$ is a majorant of the square root of $|\mathrm{d}^2u/\mathrm{d}x^2|$ and a constant, namely,

$$\sqrt{|\mathrm{d}^2u/\mathrm{d}x^2|} + 1 \le \psi(x,\varepsilon) \ , \qquad 0 \le x \le 1 \ , \tag{30}$$

where $\psi(x,\varepsilon)$ satisfies (28), and consequently

$$\int_0^1 (\sqrt{|\mathrm{d}^2u/\mathrm{d}x^2|} + 1)\mathrm{d}x \le M \ .$$

In this case,

$$\Big|\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}[x(\xi,\varepsilon)]\Big(\frac{\mathrm{d}x}{\mathrm{d}\xi}(\xi,\varepsilon)\Big)^2\Big| \le M \ , \qquad 0 \le \xi \le 1 \ ,$$

and so the first item of (29) is really uniformly bounded.

Since from (27) for $l_0 = 0,\ l_1 = 1$ we get

$$\frac{\mathrm{d}x}{\mathrm{d}\xi}(\xi,\varepsilon) = c/\psi(x(\xi,\varepsilon),\varepsilon),\ x(0,\varepsilon) = 0,\ x(1,\varepsilon) = 1,\ 0 \le \xi \le 1,\ c = \int_0^1 \psi(x,\varepsilon)\mathrm{d}x,$$

therefore

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\xi^2}(\xi,\varepsilon) = -\frac{c^2}{\psi^3[x(\xi,\varepsilon),\varepsilon]}\frac{\mathrm{d}\psi}{\mathrm{d}x}[x(\xi,\varepsilon),\varepsilon] \ , \qquad 0 \le \xi \le 1. \tag{31}$$

However, relying on (30) for the function $u(x,\varepsilon) = \exp(-x/\varepsilon) - x$, having an exponential boundary layer, we can assume $\psi(\xi,\varepsilon) = 1 + \varepsilon^{-1}\exp(-x/(2\varepsilon))$, and so, from (31)

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\xi^2} = \frac{c^2\varepsilon^{-2}\exp(-x/(2\varepsilon))}{2(1+\varepsilon^{-1}\exp(-x/(2\varepsilon)))^3} = \frac{c^2}{16\varepsilon} \ ,$$

when $x = -2\varepsilon \ln \varepsilon$. Thus, in order to generate an explicit coordinate transformation $x(\xi, \varepsilon)$ providing (29), it is not sufficient to use in (27) a function $\psi(x, \varepsilon)$ satisfying (30). It appears that this function $\psi(x, \varepsilon)$ should satisfy (28) and

$$(|\mathrm{d}^4 u/\mathrm{d}x^4|)^{1/4} + 1 \le \psi(x, \varepsilon),\ 0 \le x \le 1.$$

Analogous statements hold for obtaining mappings $x(\xi, \epsilon)$ that eliminate arbitrary layers up to $n \ge 1$ using (27). Specifically, the function $\psi(x, \epsilon)$ must satisfy (28) and the condition

$$|u^{(j)}(x, \varepsilon)|^{1/j} + 1 \le \psi(x, \varepsilon)\ , \quad j = n^2\ , \quad 0 \le x \le 1\ ,$$

(see [6]). Some procedures for generating such transformations are described in [6] and [41].

3.3 Explicit Transformations

The vicinity of each critical point x_0 (the centre of a layer) can be divided into intervals such that the function $\psi(x, \varepsilon)$ in equation (27) is majorised on each of the intervals by a corresponding elementary function with which the equation is explicitly integrated, producing the local transformation $x(\xi, x_0, \varepsilon)$. By matching these local transformations with themselves and with polynomial functions, a global explicit transformation $x(\xi, \varepsilon) : [0, 1] \to [l_0, l_1]$ can be obtained, eliminating singularities of the solution on the whole interval $[l_0, l_1]$. However, if the physical problem has too many critical points, this process of generating global transformations that eliminate the singularities of a solution may seem too laborious and inefficient. In this case, a simpler and more reliable way to perform practical calculations is to compute the transformations by solving the numerical problem (27) with a suitable weighting function $\psi(x, \varepsilon)$, typically defined on the whole interval $[l_0, l_1]$ by estimates of the solution derivatives as

$$\psi(x, \varepsilon) = |u^{(j)}(x, \varepsilon)|^{1/j} + 1,$$

see for example [6], [15], and [55].

3.3.1 Transformations eliminating exponential layers

In order to eliminate locally (in the vicinity of a boundary layer) exponential singularity (2) with $x_0 = 0$, i.e.,

$$|\ u^{(i)}(x, \varepsilon)\ | \leqslant M[\varepsilon^{-ik} \exp(-bx/\varepsilon^k) + 1]\ , \quad b > 0\ , \quad 1 \leqslant i \leqslant n\ , \quad 0 \le x \le m\ , \qquad (32)$$

of a function $u(x, \varepsilon)$ up to order n in a new coordinate ξ, a popular basic logarithmic contraction function $x_1(\xi, \varepsilon, a, k)$ can be used:

$$x_1(\xi, \varepsilon, a, k) = -\frac{\varepsilon^k}{a} \ln(1 - d\xi)\ , \quad d = \frac{1 - \varepsilon^{k/n}}{\xi_1^n}\ , \quad 0 \le \xi \le \xi_1^n\ , \qquad (33)$$

with the restriction $b/n^2 \ge a > 0$, and then it should be prolongated smoothly on the interval $[0, m_1]$ (see [4], p. 55 and [6], pp. 181-183). This type of local transformation was introduced by Bakhvalov [1].

Another popular piecewise uniform transformation

$$x(\xi,\varepsilon,b)=\begin{cases}2\sigma\xi\ , & 0\le\xi\le 1/2\ ,\\ \sigma+2(1-\sigma)\xi\ , & 1/2\le\xi\le 1\ ,\end{cases} \tag{34}$$

where $\sigma = \min\{0.5,(n/b)\varepsilon\ln N\}$, proposed by Shishkin [2] for generating grids in exponential layers, is also dependent on constant b in (32), so these grids with a fixed constant will not be suitable for all $b\in(0,\infty)$ in (2). Compared with the grid of Bakhvalov, the grid of Shishkin, though contrived later, provides numerical solutions with $\varepsilon-$uniform errors of lower accuracy.

3.3.2 Transformations Eliminating both Exponential and Power-of-Type-1 Layers

A coordinate transformation, designated as $x_{p1}(\xi,\varepsilon,p,k)$ of the class $C^l[0,1]$, $l\le n$, described in [6], pp. 183-185 and in [4], pp. 53-55, for eliminating power-of-type-1 singularities $\varepsilon^{k\alpha}/(\varepsilon^k+x)^{\alpha+i}$, $0\le x\le 1$ of scale k up to order n, has the following form:

$$x_{p1}(\xi,\varepsilon,p,k)=\begin{cases}c\varepsilon^k((1-d\xi)^{-1/p}-1)\ , & 0\le\xi\le\xi_0^n\ ,\\ c\Big[\varepsilon^{k(1-v/p)}-\varepsilon^k+\Big(\dfrac{\varepsilon^k}{(1-d\xi)^{1/p}}\Big)'(\xi_0^n)(\xi-\xi_0^n)+ & \\ +\dfrac{1}{2}\Big(\dfrac{\varepsilon^k}{(1-d\xi)^{1/p}}\Big)''(\xi_0^n)(\xi-\xi_0^n)^2+\ldots+ & \\ +\dfrac{1}{l!}\Big(\dfrac{\varepsilon^k}{(1-d\xi)^{1/p}}\Big)^{(l)}(\xi_0^n)(\xi-\xi_0^n)^l+c_0(\xi-\xi_0^n)^{l+1}\Big]\ , & \xi_0^n\le\xi\le 1\ ,\end{cases} \tag{35}$$

where $d=(1-\varepsilon^{kv})/\xi_0^n$; $v=p/(1+np)$; p is an arbitrary positive constant satisfying $0<p\le\alpha/n^2$; $1>\xi_0^n$ (for example $\xi_0^n=1/2$); c_0 is an arbitrary positive constant; $c>0$ is such that the necessary boundary condition $x_{p1}(1,\varepsilon,p,k)=1$ is satisfied ($c<1/(c_0(1-\xi_0^n)^{l+1})$), ($c<1/(c_0(1-\xi_0^n)^{l+1})$); $l\le n$;

$$\left(\frac{\varepsilon^k}{(1-d\xi)^{1/p}}\right)^{(i)}(\xi_0^n)=d^i\frac{1}{p}\left(\frac{1}{p}+1\right)\cdots\left(\frac{1}{p}+i-1\right)\varepsilon^{kp(n-i)/(1+np)},\quad i\ge 1.$$

In particular, for $l=0$ the transformation is as follows:

$$x_{p1}(\xi,\varepsilon,p,k)=\begin{cases}c\varepsilon^k((1-d\xi)^{-1/p}-1)\ , & 0\le\xi\le\xi_0^n\ ,\\ c\Big[\varepsilon^{kpn/(1+np)}-\varepsilon^k+c_0(\xi-\xi_0^n)\Big], & \xi_0^n\le\xi\le 1\ .\end{cases}$$

For example, transformations (35) with $k=1$ eliminate the singularity $\varepsilon^\alpha/(\varepsilon+x)^{\alpha+i}$, $1\le i\le n$. By estimation (14), this transformation, with an arbitrary $p>0$ and $0<v\le p/(1+np)$, eliminates the exponential singularity $(1/\varepsilon^{ik})\exp(-mx/\varepsilon^k)$ up to an arbitrary order n (see [4], pp. 54-55). Thus, transformation (35) is more convenient for eliminating exponential singularities (32) than transformations (33) or (34) dependent on b, since the constant p in (35) can be arbitrary regardless of the constant b in (32), which in practical problems modelled for example by Navier-Stokes equations typically is unknown; therefore, with an arbitrary fixed constant $p>0$, this transformation alone is valid for all constants $b \in (0, \infty)$ in (32) for eliminating exponential singularities of $u(x, \varepsilon)$ up to order n. A simpler form of transformation (35) with $p = 1$ was originally published in [46], while with an arbitrary $p > 0$ in [56], [57], and [43]. The grid obtained through transformation (35) is the most effective for numerical modelling of viscous flows over a plate [5] and in a cavern [4], compared with results obtained on other known grids.

3.3.3 Transformations eliminating power-of-type-2 layers

A coordinate transformation, designated as $x_{p2}(\xi, \varepsilon, t, k)$, for eliminating power-of-second-type singularities $(\varepsilon^k + x)^{b-i}$, $0 \le x \le 1$, up to order n, described in [6], has the following form:

$$x_{p2}(\xi, \varepsilon, t, k) = \frac{(\varepsilon^{kt} + \xi)^{1/t} - \varepsilon^k}{(\varepsilon^{kt} + 1)^{1/t} - \varepsilon^k} , \quad 0 \le \xi \le 1 , \tag{36}$$

where $0 < t \le \min\{b/n, 1/n\}$. Notice that when $b \ge n$, the function $(\varepsilon + x)^{b-n}$ is ε-uniformly bounded, so that one can consider this singularity only for $0 < b < n$. This transformation was originally published in [56] and [43]. Theoretical results regarding uniform convergence, and numerical experiments with a grid generated through this transformation for solving a problem (6) having a single attracting interior turning point x_0 approximated by an upwind scheme, were demonstrated in [3], [6] (Sect. 8.1.6, example 12), [4], and also also in [58], where high–order uniform convergence in an integral norm was proved for a FEM.

3.3.4 Transformations eliminating logarithmic layers

Solution derivatives near the boundary point $x_0 = 0$ can also be estimated by the majorant

$$|u^{(i)}(x, \varepsilon)| \le M[1 + 1/((\varepsilon^k + x)^i |\ln \varepsilon|) , \quad 1 \le i \le n , \quad 0 \le x \le m , \tag{37}$$

see (20). Unfortunately, the transformation which would eliminate this singularity up to order $n > 1$ has not yet been found. The following transformation eliminates this singularity only up to order 1::

$$x_l(\xi, \varepsilon, k) = \begin{cases} c\varepsilon^k \Big[\Big(1 + \dfrac{1}{\varepsilon^k \ln(\varepsilon^{-1})}\Big)^{\xi/\xi_0} - 1\Big] , \quad 0 \le \xi \le \xi_0 , \\ c\Big[\ln^{-1}(\varepsilon^{-1}) + \dfrac{1}{\xi_0}(\varepsilon^k + \ln^{-1}(\varepsilon^{-1})) \ln\Big(1 + \dfrac{1}{\varepsilon^k \ln(\varepsilon^{-1})}\Big)(\xi - \xi_0) + \\ c_0(\xi - \xi_0)^2\Big] , \quad \xi_0 \le \xi \le m_1 . \end{cases} \tag{38}$$

Thus, the problem of finding a coordinate transformation eliminating singularity (20) up to order $n > 1$ yet remains.

Coordinate transformation (38) has some relation to a function introduced by Eriksson [59]:

$$x_{Er}(\xi) = (e^{d\xi} - 1)/(e^d - 1), \qquad d > 0 , \qquad 0 \le \xi \le 1 ,$$

popular in CFD for providing layer resolution near the boundary $x = 0$. Namely, if d in the Eriksson function is equal to $\ln(1 + 1/\varepsilon^{-k} \ln(\varepsilon^{-1}))$ then it coincides with the contraction transformation (38) when $\xi_0 = m_1 = 1$. This relation shows clearly how to adjust the grid spacing automatically to the physical small parameter ε by means of the Eriksson function. Other functions, based on the inverse hyperbolic sines and tangents, were introduced by Vinokur [60] to treat exponential singularities.

3.3.5 Transformations eliminating hybrid layers

To eliminate a hybrid singularity, which is a combination of power-of-first- and power-of-second-type singularities of scale k:

$$\varepsilon^{k\alpha}/(\varepsilon^k + x)^{\alpha+i} + (\varepsilon^k + x)^{b-i}\ , \quad 0 \le x \le 1; \tag{39}$$

up to order n we use a composition of two coordinate transformations (36) and (35), one of which eliminates power-of-type-2 layers and the other eliminates power-of-type-1 layers. It was proved in [4], that hybrid singularities (39) for $0 \le x \le 1$ is eliminated up to n by the coordinate transformation designated as $x_h(\xi, \varepsilon, \alpha, b, p, k)$:

$$\begin{gathered} x_h(\xi, \varepsilon, \alpha, b, p, k) = \frac{(\varepsilon^{kt} + x_{p1}(\xi, \varepsilon, p, kt))^{1/t} - \varepsilon^k}{(\varepsilon^{kt} + 1)^{1/t} - \varepsilon^k}\ , \\ 0 < t \le \min\{b/n,\ 1/n\}\ , \quad 0 < p \le \alpha/(tn^2)\ , \quad 0 \le \xi \le 1\ . \end{gathered} \tag{40}$$

Transformation (40) is suitable to eliminate not only hybrid singularity (39) but also eliminates either power-of-second-type or power-of-type-1 singularities.

A layer-resolving grid to problem (1) with an interior turning point $x_0 = 0$ is defined through a piece-wise mapping $x(\xi, \varepsilon, \alpha, b, p, 1/2) : [-1, 1] \to [-1, 1]$, using (40), in the following form:

$$x(\xi, \varepsilon, \alpha, b, p, 1/2) = \begin{cases} -x_h(-\xi, \varepsilon, \alpha, b, p, 1/2), & \xi \in [-1, 0], \\ x_h(\xi, \varepsilon, \alpha, b, p, 1/2), & \xi \in [0, 1]. \end{cases} \tag{41}$$

For a problem with an arbitrary interior turning point x_0 in the interval $[l_0, l_1]$, one can use an additional monotone function $\varphi_{x_0}(x)$, which maps the interval $[-1,\ 1]$ onto $[l_0, l_1]$ with the restrictions $\varphi_{x_0}(-1) = l_0$, $\varphi_{x_0}(0) = x_0$, $\varphi_{x_0}(1) = l_1$. The corresponding transformation for generating layer-resolving grids is defined as a composition of $x(\xi, \varepsilon, \alpha, b, p, 1/2)$ and $\varphi_{x_0}(x)$ (see [6], Sect. 8.1.6).

4. TURNING-POINT PROBLEMS

Turning-point problems are considered important in practical applications and frequently involve layers that are non-exponential. Some examples of turning-point problems and analytical and numerical aspects for their study are discussed in the reviews: [14], [18] and the books: [3], [6], [34],[41]. The analytical and numerical treatments of turning-point problems are more difficult than of non-turning-point problems. In particular, it is not always possible to decompose the solution into regular and singular components to find estimates of the solution derivatives, since the reduced ($\varepsilon = 0$) problem may be ill-posed, with the coefficient before the higher derivative of the reduced problem being zero at some point. Thus, the values of the regular component and/or its derivatives may be unbounded, even though the reduced problem is independent of ε. Some special techniques for obtaining estimates of the derivatives of solutions to turning-point problems are shown in [42], [3], [29], [6],[27], and in Sect. 4.1 of this paper.

4.0.1 Stationary Problems

Popular stationary turning-point problems have the form of (1), namely:

$$-(\varepsilon + d(x))^{\nu} u'' + a(x)u' + f(x,u) = 0,\ l_0 < x < l_1,\ u(l_0) = A_0,\ u(l_1) = A_1, \tag{42}$$

where $d(x) \geq 0,\ \nu \geq 1,\ f_u(x,u) > 0,\ a(x_0) = 0$ at some point $x_0,\ l_0 \leq x_0 \leq l_1$ (see [3], Sect. 3.4.2 and [30]). A solution $u(x,\varepsilon)$ to a particular simple case of problem (42)

$$-\varepsilon u'' + axu' + u = 0, \quad 0 < x < 1, \quad u(0) = A_0,\ , \quad u(1) = A_1, \tag{43}$$

has a single power-of-type-1 boundary layer of scale $k = 1/2$ near $x = 0$ when $a > 0$, while having two boundary layers when $a < 0$: a hybrid layer of scale $k = 1/2$ near $x = 0$ and an exponential layer of scale $k = 1$ near $x = 1$ (see [3], Sect. 3). A solution $u(x)$ to the reduced problem ($\varepsilon = 0$)

$$axu' + u = 0, \quad 0 < x < 1, \quad u(1) = A_1,$$

which is independent of the small parameter ε, is $u(x) = A_1 x^{-1/a}$ in the vicinity of the point $x = 0$ for $a > 0$, i.e., $u^i(x) \to \infty$, $i \geq 0$, when $a > 0$, $x \to 0$; while in the vicinity of the point $x = 0$ for $a < 0$, we may consider the reduced problem for $0 < x < 1/2$, $u(1/2) = u(1/2,\varepsilon)$, whose solution is $u(x) = u(1/2,\varepsilon)x^{-1/a}$, where $u(x,\varepsilon)$ is a solution to (43). Thus, in this case, $u^i(x) \to \infty$, for $i > 0$, when $-a > 1$, $x \to 0$, although the reduced equation is independent of the small parameter ε. Therefore, it is not possible to decompose the solution of problem (42) into regular and singular components for finding estimates of solution derivatives.

The same remarks are obvious in the case of a turning point and variable- diffusion coefficient in (42), e.g., $\varepsilon + x^i,\ i \geq 1$, since then the reduced problem is also a two-point boundary-value problem for whose solution $u(x)$ we get $\max_x u^{(i)}(x) = \max_x(\lim_{\varepsilon\to+0} u^{(i)}(x,\varepsilon)) = \infty$, where $u(x,\varepsilon)$ is a solution of the original problem having a layer.

Certain approaches are presented in [42], [46], [47], [34], [30], [29], [3], [44], and [27], which avoid decomposition into regular and singular components for analytical analyses of qualitative properties of solutions, in particular for obtaining estimates of solution derivatives for stationary problems (42) with boundary and interior turning points. In particular, approaches in [3], and [27] include a technique for obtaining more accurate estimates of $f(x,u)$ in (42) than $|f(x,u)| \leq M$, as an indispensable tool for obtaining estimates of solution derivatives in the case of a simple turning point x_0 ($a'(x_0) \neq 0$). It has been proven using this technique that solutions to stationary singularly perturbed problems (42) with $d(x) = 0$ near a boundary multiple turning point x_0 ($a'(x_0) = 0$) have exponential-type layers (see [51], [3], [29]), while solutions with a simple boundary turning point have either power-of-1 or hybrid layers, depending on whether the turning point is attractive or repulsive (see[3]). It was also shown in [27] and [41] using this technique that solutions to the problem (42) with a turning point $x_0 = 0$, ($a'(0) \neq -1$) and $d(x) = x^2,\ \nu = 1$ have hybrid layers. The case $a'(0) = -1$ remains to be investigated, but perhaps the layer is logarithmic, similar to solutions of (42) with $d(x) = x$, $\nu = 1$, $l_0 = 0$, $a(0) = -1$ having a boundary logarithmic layer of scale $k = 1$ near $x = 0$ (see [3], section 3.4).

4.0.2 Evolutionary Problems

The same remarks apply for non-stationary problems with turning points:

$$-(\varepsilon + d(x))\frac{\partial^2 u(x,t)}{\partial x^2} + a(x,t)\frac{\partial u(x,t)}{\partial x} + b(x,t)u(x,t) + d(x,t)\frac{\partial u(x,t)}{\partial t} + f(x,t) = 0,$$
$$u(x,0) = u_0(x), \quad -l_0 \le x \le l_1, \quad u(l_0,t) = \alpha_1(t), \quad u(l_1,t) = \alpha_2(t), \quad t \in [0,T], \tag{44}$$

where $(x,t) \in (l_0,l_1) \times [0,T]$. The functions $a(x,t)$, $b(x,t)$, $d(x,t)$, $f(x,t)$ and initial condition $u_0(x)$ are sufficiently smooth and $d(x,t) \ge \delta > 0$ in $(l_0,l_1) \times (0,T]$. Furthermore, (i) $a(x_0,t) = 0$, $t \in [0,T]$, (ii) $b(x,t) \ge b > 0$, $(x,t) \in (l_0,l_1) \times (0,T]$.

It is noteworthy that if all coefficients within equation (44) as well as the boundary conditions are stationary, i.e., independent of the variable t, then its solution tends to the solution of the corresponding stationary problem as time increases. Consequently, it seems likely that the layers of solutions to parabolic problems with turning points and sufficiently smooth and compatible initial and boundary data are comparable to the layers of solutions to the corresponding steady-state problems. For example, when $d(x) = 0$ and there is a multiple boundary turning point, the layers are exponential; whereas when there is a simple boundary turning point, they are either of power-of-type-1 or hybrid layers. However, some authors of papers inaccurately draw the conclusion that there are only exponential layers in evolutionary problems with turning points and $d(x) = 0$, regardless of whether the boundary turning point is attractive, repelling, or multiple, and even if it is an interior turning point; see for example [50] for a boundary turning point and [28], [48], and [49] for an interior turning point. The conclusions are based on the erroneous assumption that the derivatives of the solutions to the reduced non-stationary problems are uniformly bounded, since they are independent of the small parameter. The authors seem to have overlooked the fact that reduced problems for equations with turning points are largely ill-posed, which distinguishes them from equations without turning points. This means that the derivatives of their solutions may not be bounded as the authors claim. Although the relationship between the parabolic problem and the steady state problem in the case of turning points remains unclear when the parabolic problem is solved over a finite time interval of T, indirect evidence can be obtained through numerical experiments. These experiments can compare the characteristics of uniformity for solutions at the x-interval for specific values of T, such as T=10, 100, 200, etc., using appropriate grids. Of course, further research is needed to fully understand this relationship.

4.1 Theoretical and Numerical Analysis of a Problem with Hybrid Layers

Numerous numerical experiments for solving non-exponential layer problems are presented in books: [41] and [6] and in papers: [27], [5], [63], [30], [44], [34], [47]. This section describes the theoretical and numerical analysis of a problem with a turning point and a variable diffusion coefficient:

$$L[u] \equiv -(\varepsilon + x)^2 u'' + a(x)u' + c(x)u = f(x) \ , \qquad 0 \le x \le 1 \ ,$$
$$\Gamma[u] \equiv [u(0,\varepsilon), u(1,\varepsilon)] = (A_0, A_1) \ , \tag{45}$$

where $1 \ge \varepsilon > 0$, $a(x),\ c(x),\ f(x) \in C^n[0,1],\ a(0) = 0,\ c(x) > 0,\ 0 \le x \le 1.$

4.1.1 Estimates of derivatives

We show here that solutions to the problem (45) exhibit a hybrid boundary layer, which is a combination of power-of-first-type and power-of-second-type layers, depending on $c(0)$ and $a = a'(0)$.

It is well known that the pair (L, Γ) in (45) is inverse monotone, i.e., if for two functions $u(x, \varepsilon)$ and $v(x, \varepsilon),\ 0 \le x \le 1$,

$$(L, \Gamma)[u] \le (L, \Gamma)[v],\ \ 0 \le x \le 1,\ \ \text{then}\ \ u(x, \varepsilon) \le v(x, \varepsilon),\ \ 0 \le x \le 1.$$

This gives ε uniform bounds on solutions $u(x, \varepsilon)$ to (45):

$$|u(x, \varepsilon)| \le M\ , \qquad 0 \le x \le 1\ . \tag{46}$$

Similar to the proof of estimate (11) in [27], the following estimate is proved:

$$|u^{(i)}(x, \varepsilon)| \le \frac{M}{(\varepsilon + x)^i}\ , \qquad n+1 \ge i \ge 0,\ 0 \le x \le 1\ . \tag{47}$$

As quantity $\int_0^1 (1/(\varepsilon + x))\mathrm{d}x = \ln(\varepsilon + 1) - \ln\varepsilon$ is not uniformly bounded, estimate (47) is not very good, since in accordance with formula (2.26) from [3] for the first derivative of a solution to (45), the following inequality is held: $\int_0^1 |u'(x, \varepsilon)|\mathrm{d}x \le M$, i.e., the variation of the solution $u(x, \varepsilon)$ on the interval [0, 1] is uniformly bounded. Therefore, for the purpose of defining layer-damping transformations $x(\xi, \varepsilon)$ satisfying (26) for $i = 1$, we have to improve estimate (47), such that $|u'(x, \varepsilon)| \le \phi(x, \varepsilon)$ and $\int_0^1 \phi(x, \varepsilon)\mathrm{d}x \le M$.

To obtain a proper estimate of $u^{(i)}(x, \varepsilon)$, more accurate than (47), we will first find necessary bounds on the function $c(x)u(x, \varepsilon) - f(x)$. It is obvious that $|c(x)u(x, \varepsilon) - f(x)| \le M$. To improve this estimate, if $u(x, \varepsilon)$ is a solution to (45), we use the preliminary estimate (47) for $i = 1$, and the operator L from (45). If $u(x, \varepsilon)$ is a solution to (45), then

$$L[cu - f](x, \varepsilon) = -(\varepsilon + x)^2[2c'(x)u'(x, \varepsilon) + c''u(x, \varepsilon) - f''(x)] + a(x)[c'(x)u(x, \varepsilon) - f'(x)] \tag{48}$$

and taking into account (46) and (47) for $i = 1$, and the condition $a(0) = 0$, we find from (48)

$$|L[cu - f](x, \varepsilon)| \le M(\varepsilon + x)\ , \qquad 0 \le x \le 1\ , \tag{49}$$

for some $M > 0$. Now, for estimating $c(x)u(x, \varepsilon) - f(x)$ we introduce the following barrier function

$$b(x, \varepsilon) = M_1(\varepsilon/(\varepsilon + x))^\alpha + M_2(\varepsilon + x)^\beta\ , \qquad \alpha > 0\ , \qquad \beta > 0\ . \tag{50}$$

We have

$$L[b](x, \varepsilon) = M_1\varepsilon^\alpha(\varepsilon + x)^{-\alpha}b_1(x, \varepsilon) + M_2(\varepsilon + x)^\beta b_2(x, \varepsilon)\ , \tag{51}$$

where

$$b_1(x, \varepsilon) = -\alpha(\alpha + 1) + c(x) - \alpha a(x)/(\varepsilon + x)\ ,$$
$$b_2(x, \varepsilon) = \beta(1 - \beta) + c(x) + \beta a(x)/(\varepsilon + x)\ .$$

Case $a + c(0) > 0$ Here and further we designate $a'(0)$ by a. If $a + c(0) > 0$, we can assume in (50) $\beta = 1$, as then

$$b_2(x, \varepsilon) = c(x) + a - \varepsilon a/(\varepsilon + x) + (a(x) - xa)/(\varepsilon + x) \geq m_0 \, , \qquad 0 \leq x \leq m_1 \, ,$$

for some $m_0 > 0$ and $m_1 > 0$, while, for a small $\alpha > 0$, namely, when $c(0) - \alpha(a + |a|)/2 - \alpha(\alpha + 1) > 0$,

$$b_1(x, \varepsilon) \geq m_2 \, , \qquad 0 \leq x \leq m_3 \, ,$$

for some $m_2 > 0$ and $m_3 > 0$. Therefore, relying on (50) with sufficiently large M_1 and M_2, we come, using (49) and (51), to

$$L[b](x, \varepsilon) \geq L[cu - f](x, \varepsilon) \geq L[-b](x, \varepsilon) \, , \qquad 0 \leq x \leq m_4 \, ,$$

for some $m_4 > 0$. Suggesting also that $M_1 \geq |c(0)u(0, \varepsilon) - f(0)|$, $M_2 \geq |c(m_4)u(m_4, \varepsilon) - f(m_4)|/(m_4)^\beta$, from the principle of inverse monotonicity of the pair $[L, \Gamma]$ in $C^2[0, m_4]$, we get, taking into account (51), that

$$|c(x)u(x, \varepsilon) - f(x)| \leq M[\varepsilon^\alpha/(\varepsilon + x)^\alpha + \varepsilon + x] \, , \qquad 0 \leq x \leq m_4 \, ,$$

and using (46), it follows

$$|c(x)u(x, \varepsilon) - f(x)| \leq M[\varepsilon^\alpha/(\varepsilon + x)^\alpha + \varepsilon + x] \, , \qquad 0 \leq x \leq 1 \, , \tag{52}$$

for some $M > 0$, if $a'(0) + c(0) > 0$ and $\alpha > 0$ is such, that $c(0) - \alpha(a + |a|)/2 - \alpha(\alpha + 1) > 0$.

Case $a + c(0) \leq 0$ Now let $a + c(0) \leq 0$, i.e., $-a \geq c(0)$, then, for an arbitrary positive β, satisfying $\beta(1 - \beta) + c(0) + \beta a'(0) > 0$, for example, $0 < \beta < \min(1, c(0)/|a|)$, we get

$$b_2(x, \varepsilon) = \beta(1 - \beta) + c(x) + \beta a(x)/(\varepsilon + x) \geq m_0 \, , \qquad 0 \leq x \leq m_1 \, ,$$

for some $m_0 > 0$. Further, similarly as proving (52), we conclude that

$$|c(x)u(x, \varepsilon) - f(x)| \leq M[\varepsilon^\alpha/(\varepsilon + x)^\alpha + (\varepsilon + x)^\beta] \, , \qquad 0 \leq x \leq 1 \, , \tag{53}$$

for $\beta > 0$ and $\beta(1 - \beta) + c(0) + \beta a > 0$, if $a + c(0) \leq 0$ and $\alpha > 0$ is such, that $c(0) - \alpha a - \alpha(\alpha + 1) > 0$.

By resolving (45) with respect to $u'(x, \varepsilon)$ we obtain

$$u'(x, \varepsilon) = u'(x_0, \varepsilon) \exp[\psi(x_0, x, \varepsilon)] + \int_{x_0}^{x} \frac{c(\xi)u(\xi, \varepsilon) - f(\xi)}{(\varepsilon + \xi)^2} \exp[\psi(\xi, x, \varepsilon)] \mathrm{d}\xi \, , \qquad 0 \leq x \leq 1 \, , \tag{54}$$

where $\psi(\xi, x, \varepsilon) = \int_{\xi}^{x} a(\eta)/(\varepsilon + \eta)^2 \mathrm{d}\eta$.

Case $a = 0$ In this case $|\psi(\xi, x, \varepsilon)| \leq M \;\; 0 \leq \xi \;, x \leq 1$, and $c(x)u(x, \varepsilon) - f(x)$ is estimated by (52). Therefore, using in (54) estimates (52) and (47) for $i = 1$ and $x_0 \geq m > 0$, we obtain

$$|u'(x, \varepsilon)| \leq M[1 + \varepsilon^\alpha/(\varepsilon + x)^{\alpha+1} + |\ln(\varepsilon + x)|] \, , \qquad 0 \leq x \leq 1 \, , \tag{55}$$

for an arbitrary $\alpha > 0$, satisfying $c(0) - \alpha(\alpha+1) > 0$. Relying on this estimate, we obtain as in (48),

$$|L[cu - f](x,\varepsilon)| \le M[\varepsilon^\alpha/(\varepsilon+x)^\alpha + (\varepsilon+x)^2\ln(\varepsilon+x)]\ , \qquad 0 \le x \le 1\ ,$$

as $a = 0$. Therefore, using the barrier function (50) with β, satisfying both $2 > \beta > 0$ and $\beta(1-\beta) + c(0) > 0$, similar to obtaining (53), we conclude that

$$|c(x)u(x,\varepsilon) - f(x)| \le M[\varepsilon^\alpha/(\varepsilon+x)^\alpha + (\varepsilon+x)^\beta]\ , \qquad 0 \le x \le 1\ , \tag{56}$$

for $a = 0$, where $\alpha > 0$ is such, that $c(0) - \alpha(\alpha+1) > 0$. Thus, in this case, we get from (54) the following, more accurate than (55), estimate:

$$|u'(x,\varepsilon)| \le M[1 + \varepsilon^\alpha/(\varepsilon+x)^{\alpha+1}]\ , \qquad 0 \le x \le 1\ . \tag{57}$$

Relying again on this estimate of $u'(x,\varepsilon)$ in (48), we readily obtain that formula (56) is valid when β satisfies both $2 \ge \beta > 0$ and $\beta(1-\beta) + c(0) > 0$. Using estimates (56) and (57), we conclude from (45), that

$$|u^{(i)}(x,\varepsilon)| \le M[1 + \varepsilon^\alpha/(\varepsilon+x)^{\alpha+i} + (\varepsilon+x)^{\beta-i}]\ , \quad i \ge 1\ , \quad 0 \le x \le 1\ , \tag{58}$$

if $a = 0$ and $\alpha > 0$ is such, that $c(0) - \alpha(\alpha+1) > 0$ and β satisfies both $2 \ge \beta > 0$ and $\beta(1-\beta) + c(0) > 0$. In particular, $\beta = 2$ if $c(0) > 2$ or $\beta = 1$ for an arbitrary $c(0) > 0$ complies with these requirements.

Case $a \ne 0$ In this case we have for $\psi(\xi,x,\varepsilon)$ in (54)

$$\psi(\xi,x,\varepsilon) = \int_\xi^x \frac{(\varepsilon+\eta)a + a(\eta) - a\eta - a\varepsilon}{(\varepsilon+\eta)^2}\mathrm{d}\eta = a\ln\Big(\frac{\varepsilon+x}{\varepsilon+\xi}\Big) + g_1(\xi,x,\varepsilon),$$

where $g_1(\xi,x,\varepsilon) = \int_\xi^x (a(\eta) - a\eta - a\varepsilon)/(\varepsilon+\eta)^2\mathrm{d}\eta$. As $|a(\eta) - a\eta| \le M\eta^2$, so

$$|g_1(\xi,x,\varepsilon)| \le M\ , \qquad 0 \le \xi, x \le 1\ , \tag{59}$$

and, therefore,

$$\exp[\psi(\xi,x,\varepsilon)] = \exp[g_1(\xi,x,\varepsilon)]\Big(\frac{\varepsilon+x}{\varepsilon+\xi}\Big)^a \le M\Big(\frac{\varepsilon+x}{\varepsilon+\xi}\Big)^a. \tag{60}$$

Thus, from (54) we get

$$u'(x,\varepsilon) = u'(x_0,\varepsilon)\Big(\frac{\varepsilon+x}{\varepsilon+x_0}\Big)^a \exp[g_1(x_0,x,\varepsilon)] + g_2(x_0,x,\varepsilon)\ , \tag{61}$$

where

$$g_2(x_0,x,\varepsilon) = (\varepsilon+x)^a \int_{x_0}^x \frac{c(\xi)u(\xi,\varepsilon) - f(\xi)}{(\varepsilon+\xi)^{2+a}}\exp[g_1(\xi,x,\varepsilon)]\mathrm{d}\xi\ .$$

Now, using estimates (52) and (53), in which $\alpha > 0$ and $\beta > 0$, and (59), we get

$$\begin{aligned} g_2(x_0,x,\varepsilon) &\le M(\varepsilon+x)^a\Big|\int_{x_0}^x \frac{\varepsilon^\alpha}{(\varepsilon+\xi)^{\alpha+2+a}}\mathrm{d}\xi + \int_{x_0}^x \frac{\mathrm{d}\xi}{(\varepsilon+\xi)^{2+a-\beta}}\Big| \le \\ &\le M_1(\varepsilon+x)^a\Big[\frac{\varepsilon^\alpha}{(\varepsilon+x)^{\alpha+1+a}} + \frac{\varepsilon^\alpha}{(\varepsilon+x_0)^{\alpha+1+a}} + \frac{1}{(\varepsilon+x)^{1+a-\beta}} + \frac{1}{(\varepsilon+x_0)^{1+a-\beta}}\Big]\ . \end{aligned} \tag{62}$$

Case $a > 0$. In this case, according to (52) $\beta = 1$ in (62), so we have from (61) and (62) with $x_0 \ge m > 0$

$$|u'(x,\varepsilon)| \le M[1 + \varepsilon^\alpha/(\varepsilon + x)^{\alpha+1}] , \qquad 0 \le x \le 1 , \tag{63}$$

for an arbitrary $\alpha > 0$, satisfying $c(0) - \alpha a'(0) - \alpha(\alpha+1) > 0$. While for an arbitrary $i \ge 1$ we get from (45) and (63)

$$|u^{(i)}(x,\varepsilon)| \le M[1 + \varepsilon^\alpha/(\varepsilon + x)^{\alpha+i} + (\varepsilon + x)^{1-i}] , \quad i \ge 1 , \quad 0 \le x \le 1 . \tag{64}$$

Case $a < 0$. Let first $-a > 1$, then, assuming $x_0 = 0$, we obtain from (61) and (62), taking into account that in accordance with (47) $u'(0) \le M\varepsilon^{-1}$,

$$|u'(x,\varepsilon)| \le M\Big[\frac{\varepsilon^{-a-1}}{(\varepsilon + x)^{-a}} + \frac{\varepsilon^\alpha}{(\varepsilon + x)^{\alpha+1}} + (\varepsilon + x)^{\beta-1}\Big] , \ -a > 1 , \ 0 \le x \le 1 , \tag{65}$$

where $\alpha > 0$ is such that $c(0) - \alpha(\alpha+1) > 0$, and β satisfies both $0 < \beta \le 1$ and $0 < \beta < c(0)/|a|$. As $\alpha = -a - 1$ satisfies the requirement $c(0) - \alpha(\alpha+1) = c(0) > 0$, so (65) yields

$$|u'(x,\varepsilon)| \le M\Big[\frac{\varepsilon^{-a-1}}{(\varepsilon + x)^{-a-1-i}} + (\varepsilon + x)^{\beta-1}\Big] , \quad -a > 1 , \quad 0 \le x \le 1 ,$$

and from (45) we readily get

$$|u^{(i)}(x,\varepsilon)| \le M\Big[\frac{\varepsilon^{-a-1}}{(\varepsilon + x)^{-a-1-i}} + (\varepsilon + x)^{\beta-i}\Big] , \quad -a > 1 , \quad i \ge 1 , \quad 0 \le x \le 1 , \tag{66}$$

for β satisfying both $0 < \beta \le 1$ and $0 < \beta < c(0)/|a'(0)|$. In particular, if $a + c(0) > 0$ we can in (65) and (66) assume $\beta = 1$.

Let now $0 < -a < 1$. Using (47), (61), and (62) with $x_0 = m > 0$, we obtain

$$|u'(x,\varepsilon)| \le M\Big[(\varepsilon + x)^a + \frac{\varepsilon^\alpha}{(\varepsilon + x)^{\alpha+1}} + \frac{1}{(\varepsilon + x)^{1-\beta}}\Big] , \qquad 0 < x \le 1 . \tag{67}$$

As $0 < -a < 1$, we can assume $\beta = 1 + a$, because in this case $\beta(1-\beta) + c(0) + \beta a = c(0) > 0$, thus estimate (53) is valid for this β, and, consequently, $(\varepsilon + x)^a = (\varepsilon + x)^{\beta-1}$, so (67) yields

$$|u'(x,\varepsilon)| \le M\Big((\varepsilon + x)^a + \frac{\varepsilon^\alpha}{(\varepsilon + x)^{\alpha+1}}\Big) , \qquad 0 < -a < 1 , \qquad 0 \le x \le 1 , \tag{68}$$

where α is an arbitrary positive number, satisfying $c(0) - \alpha(\alpha+1) > 0$. From this estimate and (45) we get

$$|u^{(i)}(x,\varepsilon)| \le M\Big[(\varepsilon + x)^{a-i+1} + \frac{\varepsilon^\alpha}{(\varepsilon + x)^{\alpha+i}}\Big] , \quad 0 < -a < 1 , \quad i \ge 1 , \quad 0 \le x \le 1 . \tag{69}$$

Summarizing the previous results we formulate the following theorem:

Theorem 2 *Let $u(x,\varepsilon)$ be a solution to* (45) *with $a = a'(0) \ne -1$. Then,*

$$|u^{(i)}(x,\varepsilon)| \le M[1 + \varepsilon^\alpha/(\varepsilon + x)^{\alpha+i} + (\varepsilon + x)^{\beta-i}] , \quad 0 \le i \le n , \quad 0 \le x \le 1 , \tag{70}$$

where
1. for $a = 0$*:* α *is an arbitrary positive number, satisfying* $c(0) - \alpha(\alpha + 1) > 0$*,* β *is an arbitrary positive number, satisfying both* $2 \geq \beta > 0$ *and* $\beta(1 - \beta) + c(0) > 0$*;*
2. for $a > 0$*:* α *is an arbitrary positive number, satisfying* $c(0) - \alpha a - \alpha(\alpha + 1) > 0$*,* $\beta = 1$*;*
3. for $0 < -a < 1$*:* α *is an arbitrary positive number, satisfying* $c(0) - \alpha(\alpha + 1) > 0$*,* $\beta = 1 + a$*;*
4. for $-a > 1$*:* $\alpha = -a - 1$*,* β *is an arbitrary positive number, satisfying both* $0 < \beta \leq 1$ *and* $0 < \beta < c(0)/|a|$.

Notice, the case $a = -1$ has not yet been unraveled.

4.1.2 Numerical Algorithm

We use as an approximation of the singularly-perturbed boundary-value problem (45) the standard up wind finite difference scheme on a nonuniform grid x_i, $i = 0, 1, \dots, N$, $x_0 = 0 < x_1 < \dots < x_N = 1$:

$$-\frac{2(\varepsilon + x_i)^2}{h_i + h_{i-1}}\Big[\frac{u^h_{i+1} - u^h_i}{h_i} - \frac{u^h_i - u^h_{i-1}}{h_{i-1}}\Big] + a_-(x_i)\frac{u^h_{i+1} - u^h_i}{h_i} +$$

$$+a_+(x_i)\frac{u^h_i - u^h_{i-1}}{h_{i-1}} + c(x_i)u_i = f(x_i)\ , \quad i = 1, 2, \dots, N-1\ , \tag{71}$$

where $u^h_0 = A_0$, $u^h_N = A_1$, $h_i = x_{i+1} - x_i$, and $a_\pm = (a \pm |a|)/2$. The nodes x_i, $i = 0, \dots, N$, of a layer-resolving grid are obtained explicitly by using layer-damping transformation (40), namely,

$$x_i = x_h(ih, \varepsilon, \alpha, \beta, p, 1/2)\ , \quad i = 0, 1, \dots, N\ , \quad h = 1/N\ .$$

The calculations of problem (45) are conducted for small values of ε, for each sequence of grids with doubled numbers of grid steps: $N_t = 2^t N_h$, $t = 0, 1, \dots$, where N_h is the number for the rough grid. Usually $N_h = 50$, $t_{\max} = 5$. The numerical solution at the ith node of the grid related to N_t, is denoted by $u^{N_t}_i$, $i = 0, 1, \dots, N_t$.

For estimating the accuracy of the numerical algorithm, the following characteristic is introduced:

$$r_{t,\varepsilon} = \max_{0 \leq i \leq N_t} |u^{N_t}_i - u^{N_{t+1}}_{2i}|\ , \quad t = 0, 1, \dots\ ,$$

In addition to this, one more characteristic

$$du_{t,\varepsilon} = \max_{0 \leq i \leq N_t} |u^{N_t}_{i+1} - u^{N_t}_i|\ , \quad i = 0, 1, \dots, N_t - 1\ ,$$

is introduced, which is related to the jump of the numerical solution in the neighboring nodes.

A characteristic $r_{t,\varepsilon}$ is applied to estimate the order of the accuracy of the numerical solution:

$$\beta_1 = \log_2(r_{t,\varepsilon}/r_{t+1,\varepsilon})\ , \ t = 0, 1, \dots\ , \tag{72}$$

and, consequently, $du_{t,\varepsilon}$ to estimate the order of the numerical solution jump in the neighboring nodes

$$\beta_3 = \log_2(du_{t,\varepsilon}/du_{t+1,\varepsilon})\ , \quad t = 0, 1, \dots\ . \tag{73}$$

Note that if a solution has neither boundary nor interior layers, then for the numerical solution of this problem through the use of a stable scheme of order l on the uniform grid $x_i = ih$ the value β_1 is close to l, while β_3 is close to 1.

Theorem 3 *Let $u_i^h, i = 0, \cdots, N$ be a numerical solutions to problem (45) by scheme (71) on the grid $x_i = x_h(ih, \varepsilon, \alpha, \beta, p, 1/2),\ i = 0, 1, \ldots, N,\ h = 1/N$, where $x_h(ih, \varepsilon, \alpha, \beta, p, 1/2)$ is defined by* (40) *for $k = 1/2$, then*

$$|u_i^h - u(x_i, \varepsilon)| \le M/N\ , \quad i = 0, \cdots, N\ .$$

This theorem is proved similarly as in ([3], Subsect. 7.4.2) and [47] for problem (4) with $a(x, u) = a(x),\ a(0) = 0,\ a'(0) < 0$, having hybrid layers.

4.1.3 Numerical Experiment

This section presents results of a numerical solution to problem (45) obtained by scheme (71) on the grid $x_i = x(i/N, \varepsilon)$, $i = 0, 1, \ldots, N$, where $x(\xi, \varepsilon) : [0, 1] \to [0, 1]$ is a coordinate transformation (40) for $l = n = 2$.

Example For a numerical experiment we consider the following problem:

$$-(\varepsilon + x)^2 - 2xu' + u = \sin(3\pi x) = 0\ , \quad 0 \le x \le 1\ ,$$
$$u(0, \varepsilon) = 0\ , \quad u(1, \varepsilon) = 1\ .$$

For this problem $a = -2$, $c(0) = 1$, so $\beta = 0.4$ and $\alpha = 1$ match the requirement 4 of theorem 2. Thus, estimate (70) is as follows:

$$|u^{(i)}(x, \varepsilon)| \le M[1 + \varepsilon/(\varepsilon + x)^{1+i} + (\varepsilon + x)^{0.4-i}]\ , \quad 0 \le i \le n\ , \quad 0 \le x \le 1.$$

t	N	$r\ hybrid$	$\beta_1\ hybrid$	$du\ hybrid$	$\beta_3\ hybrid$
2	100	0.015582	1.580887	0.087672	0.935526
3	200	0.008472	0.879090	0.044250	0.986435
4	400	0.004414	0.940606	0.022439	0.979697
5	800	0.002294	0.944044	0.011287	0.991379

Table 1: shows the values of the characteristics β_1 and β_3 for $\varepsilon = 10^{-5}$, calculated using difference scheme (71) on the grid $x_i = x(i/N, \varepsilon)$, $i = 0, 1, \ldots, N$, where $x(\xi, \varepsilon) : [0, 1] \to [0, 1]$ is a coordinate transformation (40) for $l = n = 2$, $k = 1$, $t = 0.2$, $p = 1.25$.

5. CONCLUSION

The goal of this paper has been to persuade readers not to restrict themselves to studying problems with exponential layers and their primitive generalizations to problems having discontinuous or

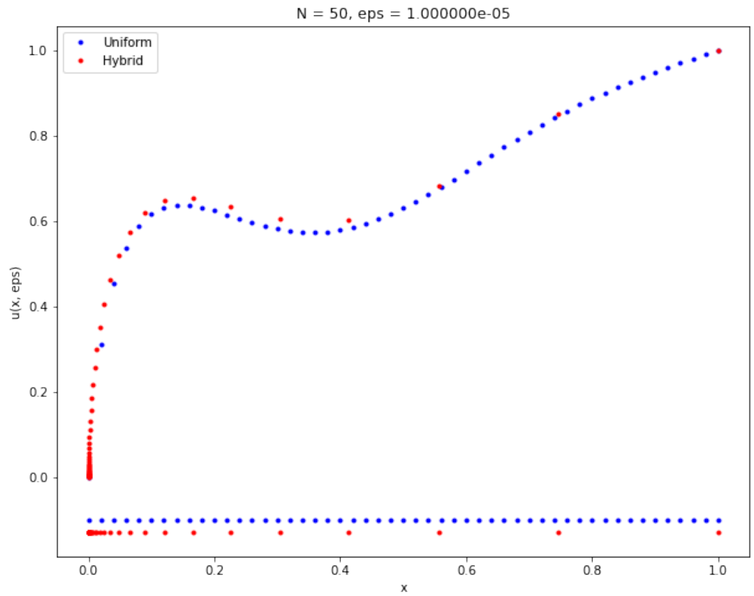

Figure 1: shows the numerical solution on the adaptive (in red) and uniform (in blue) grids.

variable-shifted coefficients in equations. The author's desire is to acquaint readers with the diversity of layer types, and to motivate them to step up efforts to study higher-level problems having such layers, thus coming closer to meeting the requirements of practical applications.

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