

What Is the Q of a Blackbody? A Small Contribution to Gustav Robert Kirchhoff's Bicentennial (1824 – 1887)

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Abstract

The blackbody spectrum “half-power points” are used to assign effective Q “quality factor” values that are found to be less than unity whether frequency or wavelength scaling is used. A comparison with values for coherent oscillators is made. This exercise blends two of Kirchhoff’s interests, and is instructive in its own right, as it bridges the often mutually exclusive engineering and scientific disciplines.

Keywords: Kirchhoff, Planck, Wien, Rayleigh, Jeans, Electrical circuits, Quality factor, Blackbody

1. INTRODUCTION

In 1860, the mathematical physicist Gustav Kirchhoff coined the term “blackbody” to denote an ideal surface that absorbs all incident electromagnetic radiation[1]. Fifteen years earlier, as a student, he had introduced the two eponymous circuit laws now universally familiar to electrical engineering (EE) students[2]^{1,2} This paper establishes another link between these disciplines with which he was

¹ On pages 513 and 514 we find the enunciation of “Student” Kirchhoff’s laws: “In order to be able to derive the given proportion in a convenient way, I will first prove the following theorem:

If galvanic currents flow through a system of wires that are connected in any way, then:

1) if the wires 1, 2, .. μ meet in one point, $I_1 + I_2 + \dots + I_\mu = 0$, where I_1, I_2, \dots denote the intensities of the currents flowing through those wires, all counted as positive towards the point of contact;

2) if the wires 1, 2, .. ν form a closed figure, $I_1 \cdot \omega_1 + I_2 \cdot \omega_2 + \dots + I_\nu \cdot \omega_\nu =$ the sum of all electromotive forces located on the path: 1, 2, .. ν ; where $\omega_1, \omega_2, \dots$ the resistances of the wires, I_1, I_2, \dots denote the intensities of the currents that flow through them, all counted as positive in *one* direction.”

² Kirchhoff was but one of many gifted “students” who contributed to a remarkable flowering of mathematics and physics in Königsberg in the 19th century. See, e.g., Arthur Ballato, “Piezoelectricity: History and new thrusts,” IEEE International Ultrasonics Symposium Proceedings (1996) 575-583. In addition to the personages noted in this paper, one might mention a few other connections. Kirchhoff was the doctoral advisor of Roland von Eötvös (PhD, 1870) and Max Noether (Dr phil., 1868). Max Noether was the father of Emmy Noether (PhD, Universität Erlangen-Nürnberg, 1907) and Fritz Noether (Dr phil., Universität München,

so conversant, and celebrates the bicentennial of his birth. Connecting blackbody physics and circuit engineering is apposite, not only for didactic reasons, but because it serves also as a reminder that not a few notable scientists began their careers with an engineering education, e.g., Röntgen, Debye, Dirac, Onsager, Bardeen, etc.

Studies of blackbody (BB) radiation led, as is very well known, to quantum mechanics, and so much has been written about it at all levels that it might be asked if anything new could, (or should!), be added.[3–35]. We answer in the affirmative. It is both pedagogically interesting and apt to apply an EE concept to quantify BB radiation. Prescinding from the legitimate objection that such a disparate application, whereby the Q concept usually applied to a coherent resonance should be applied to a completely incoherent photon fluid, we nevertheless find the result to be fully consonant with the exceedingly low values intuitively expected, but unexpectedly to be independent of the blackbody temperature.³

We begin by discussing briefly lumped electrical circuits, and introduce the concept of Q from an EE circuit point of view. This is followed by a bare-bones sketch of BB. Finally, the equivalent Qs of BB spectra are evaluated.

2. LUMPED ELECTRICAL CIRCUITS – EE 101

It can be said that with the introduction of electrostatic generators to produce electric charge, and Leiden jars for its storage, the science of electricity began its steps to maturity. Volta’s batteries [36], subsequently permitted production of steady electrical currents, and led to the laws of Ørsted,[37], Ampère,[38–40], Ohm[41], and, of course, Kirchhoff. These lumped circuit developments preceded the magisterial unification of electrodynamics at the hands of Faraday and Maxwell with the introduction of field concepts.[42–46]. Today, the historical roles are reversed, and lumped circuits are considered to follow logically from the more general concept of fields[47]. The correspondences between the Maxwellian (field) and the EE (lumped) approaches have been discussed by Hansen[48], Dicke[49], and particularly by Fano, Chu, and Adler, [50] as well as by others[51–54]. Lumped circuit elements (capacitors, inductors, and resistors)[55], represented by graphical symbols, and their mutual attachments to form circuits, follow from the assumption of unbounded lightspeed ($1/c \rightarrow 0$), and as such they have no spatial extensions, nor do they mirror the physical geometry of the system represented. They are idealized repositories of electric and magnetic energy and an element of dissipation, with associated equations: $i = C \cdot (de/dt)$ (capacitor), $e = L \cdot (di/dt)$ (inductor), and $e = R \cdot i$ (resistor). In these equations, i is the current through the element, e is the voltage across the element, C the capacitance, L the inductance, R the resistance, and d/dt the time operator.

1909). Emmy discovered deep connections between symmetries in nature and conservation laws. See E. Noether, “Invariante Variationsprobleme,” *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen. Mathematisch-Physikalische Klasse.* (1918) 235–257. Emmy emigrated to the US and Fritz emigrated to the USSR; he was shot by Stalin’s NKVD in 1941.

³ Discovery of the cosmic microwave background spectrum [Robert H. Dicke, Phillip James Edwin Peebles, Peter G. Roll, and David T. Wilkinson, “Cosmic black-body radiation,” *Astrophysical Journal* **142**(1) (1965) 414-418; Arno Allan Penzias and Robert Woodrow Wilson, “A measurement of excess antenna temperature at 4080 Mc/s,” *Astrophysical Journal* **142**(1) (1965) 419-421] led to finding the purest example of a blackbody in nature, revealed in the results of the COBE experiment. See, e.g., John C. Mather, et al., “Measurement of the cosmic microwave background spectrum by the COBE FIRAS instrument,” *Astrophysical Journal* **420**(2) (1994) 439-444; Dale J. Fixsen, et al., “The cosmic microwave background spectrum from the full COBE1 FIRAS data set,” *Astrophysical Journal* **473**(2) (1996) 576-587. Additional blackbody spectra arise from neutron star collisions, see, e.g., Albert Sneppen, “On the blackbody spectrum of kilonovae,” *Astrophysical Journal* **955**(1) (2023) 44. Blackbody radiation is truly ubiquitous; what could be more ubiquitous than the universe?

Nature proceeds in the time domain, and its descriptions take the form of differential equations (DEs). Not infrequently it is more readily interpreted by transformations into the frequency domain, whereby DEs are converted to algebraic functions. In the case of electrical networks, the DEs characterize generalized Ohm's laws relating voltage (e) across, and current (i) through an element, and the Laplace transform is used to convert these DEs of individual circuit elements into algebraic functions relating the independent and dependent variables, i.e., the input-output relations.⁴ The forerunner of this method, "operational calculus," was largely developed by Heaviside[56–63].

These individual circuit elements are then combined, in accordance with the topology of the given network, to produce network input-output relations that take the form of rational functions, i.e., quotients of finite polynomials – a form more easily manipulated and interpreted than the original DEs.^{5,6} The boundary conditions on the DEs are simply Kirchhoff's laws imposing constraints on currents at each junction of elements (nodes), and on voltages around each complete loop of elements. Circuit theory is now a mature field, having come a long way from its infancy with Kirchhoff[64–70].

3. THE CONCEPT OF Q AS A QUALITY FACTOR

The lumped circuit elements L and C store magnetic and electric energy, respectively, while R, representing loss, dissipates energy. In circuit configurations comprised of R, L, and C elements, it is found that the response to steady-state excitation varies with frequency. At particular frequencies, corresponding to solutions of the homogeneous DEs characterizing the configuration (complementary DE solutions), the responses reach local extrema, limited only by the presence of loss. Quantifying these "resonances" in magnitude and frequency extent led to the concept of circuit Qs. Johnson [71], first used the symbol "Q", while Legg and Given [72], coined the term "quality factor." Green [73], gives an excellent account of its early history. A search for an alternative etymology of "Q" did not succeed[74–85].

Equivalent definitions of Q as a selectivity parameter quantifying the sharpness of resonances appear in the EE literature[6, 48, 50, 86, 87]. Some follow from various energy or power relations, e.g. $Q = \omega \cdot ((E_{\text{electric}} + E_{\text{magnetic}}) / P_{\text{dissipated}})$; $Q = \omega \cdot (\text{Peak stored energy} / \text{Average power})$; $Q \propto (\text{Energy stored} / \text{Energy dissipated per cycle})$; or as frequency derivatives of immittance functions:

$Q \propto \omega \cdot \partial (\text{susceptance}) / \partial \omega \propto \omega \cdot \partial (\text{reactance}) / \partial \omega$; or as the shape factor of a resonance curve: $Q = f_0 / \Delta f$; or simply as a circuit relation: $Q = (1/R) \cdot \sqrt{L/C}$. [48, 88] Alternatively, it may be defined as $Q = \pi / \delta$, where δ is the logarithmic decrement when the circuit is subjected to a transient excitation in the time domain. Additional aspects are discussed by Feld [89], and

⁴ The input-output relations, for lumped, linear, finite, passive, and bilateral elements are characterized by quotients called "immittances," i.e., either impedances or admittances, defined as the ratio of an effect to its cause. For example, Ohm's law, (a zeroth-order linear DE), states that the current (i , the effect) through a resistor is proportional to the voltage (e , the cause) applied across it: $i/e = g$, the admittance.

⁵ In the sinusoidal steady-state, transients, (complementary DE solutions), are absent because of the inevitable presence of loss.

⁶ For resonances in piezoelectric materials, caution is advised, as there is an interplay between loss and piezocoupling in determining resonance widths, and thus Q. See Arthur Ballato, "Interpreting piezoceramic impedance measurements," in *Dielectric Materials and Devices*, edited by K. M. Nair, et al., (Westerville, Ohio, American Ceramic Society, 2002) 369-410.

particularly by Ohira [90, 91], and references therein. The concept of Q is also relevant in connection with probability distribution functions; see TABLE 6.

3.1 Application to the Series RLC Circuit

In the case where lumped elements R , L , and C are in series as in FIGURE 1 (a), an impressed voltage e , with $\exp(j\omega t)$ temporal variation, placed across the end terminals, will produce a steady-state common current i having the same temporal variation, albeit generally with a different phase with respect to that of the voltage. The complex input impedance is defined to be $Z = e/i$; complex admittance is defined as $Y = 1/Z$ (the term “immittance” is used as a general term for either Z or Y). While the R , L , and C values are constants, the immittance is a function of frequency, as also, in general, are its real and imaginary parts. These are defined as: resistance = $\text{Re}(Z)$, reactance = $\text{Im}(Z)$, conductance = $\text{Re}(Y)$, and susceptance = $\text{Im}(Y)$. With the further definitions of normalized frequency Ω , and quality factor Q as: $\Omega = \omega \cdot \sqrt{L \cdot C}$ and $Q = (1/R) \cdot \sqrt{L/C}$, the Laplace-transformed expressions for input impedance, normalized to R , is $Z/R = [1 + j \cdot Q \cdot (\Omega - 1/\Omega)]$. [92] It then follows that normalized conductance, $\text{Re}(R \cdot Y) = g = 1/[1 + Q^2 \cdot (\Omega - 1/\Omega)^2]$. This is plotted in FIGURE 1 (b) as function of Ω for a number of Q values[92]. The dissipated power is proportional to g , and is a maximum at $\Omega = 1$, the resonance frequency; this is the eigenvalue of the corresponding DE for the circuit, now appearing as the root of a polynomial.

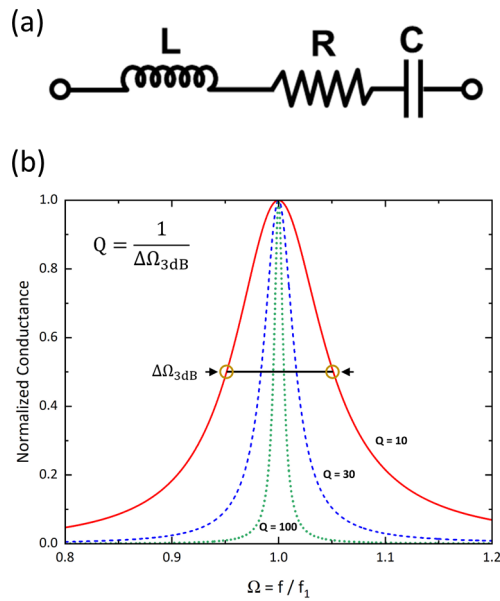


Figure 1: (a) The series RLC circuit. Resistor R is the dissipative component, evolving power (heat). Inductor L and capacitor C are lossless components that store energy. (b) Resonance curve of the series RLC circuit, with 3dB points shown. Frequency is defined as $f = \omega/2\pi$. Resonance frequency f_1 equals $\omega_1/2\pi$, where $L \cdot C \cdot \omega_1^2 \equiv 1$, and normalized frequency is $\Omega = \omega \cdot \sqrt{L \cdot C} = f/f_1$. Quality factor, $Q = (1/R) \cdot \sqrt{L/C} = 1/\Delta\Omega_{3dB}$. When discussing BB, the EE symbol “ f ” is replaced by “ ν .” As a sop to the EEs, shown are the “3dB points,” rather than the true $1/2$ -power points. These are related by $10 \cdot \log_{10}(2) \approx 3.0103$ dB.

3.2 The Half-Power Points

The frequency expanse, centered about the resonance peak, that is consonant with the definitions of Q given above, is shown in FIGURE 1 (b) as $\Delta\Omega_{3dB}$. This is the 3-dB “bandwidth” found in the EE literature. More accurately, the bandwidth is reckoned as the frequency difference between the two points where the dissipated power in the circuit is one-half of the maximum[93]. The half-power (physics) and 3-dB (EE) points are related by $10 \cdot \log_{10}(2^{\pm 1}) \approx \pm 3.0103\text{dB}$. For the series (or parallel) RLC circuit, the half-power points are determined from the condition $g = 1/2$, yielding two frequencies $\Omega^{(\pm)} = \sqrt{(1 + 1/(2 \cdot Q)^2)} \pm 1/2Q$. These are related by the usual definition: $Q = 1/[\Omega^{(+)} - \Omega^{(-)}] = 1/\Delta\Omega = (1/R) \cdot \sqrt{L/C}$.

3.3 The Butterworth-Van Dyke (Bvd) Circuit

While the RLC example is useful as an introduction to circuits, a simple modification, known as the BVD circuit, has many more applications[88, 93]. The BVD circuit consists of the series RLC circuit shunted by a parallel capacitor, C_0 . As there is no dissipation due to C_0 the expressions for $\Omega^{(\pm)}$ are unaltered. The BVD circuit is used to represent many single resonance phenomena[93]. There are two capacitors in the BVD circuit, so the quantity $r = C_0/C$, comes into play. While the quantity $g = 1/[1 + Q^2 \cdot (\Omega - 1/\Omega)^2]$ does not involve r , so that $Q = 1/[\Omega^{(+)} - \Omega^{(-)}]$ may be used, many of the other network functions, such as admittance magnitude, do, and care must be exercised in using alternative definitions of Q as equal to $f/\Delta f$ because “3 dB bandwidth” (Δf) becomes a property jointly of Q and r [94]. The ratio $1/r$ appears as a lossless coupling factor in many guises, for example in piezoelectrics and in the Lyddane-Sachs-Teller (LST) relation[93]. In these cases, the “bandwidth” arises from the coupling, and not the loss, and $1/Q = 0$.

3.4 Quality Factors of Other Phenomena

Resonance spectra having loss mechanisms are never infinitely narrow (Dirac δ -functions) but have finite bandwidths. Q , as a measure of the “quality” of the resonance, has been used in applications that vary from laser and molecular resonances to electrical circuits to electromechanical quartz oscillators to the bouncing of a golf ball on a hard surface. Some examples are shown in TABLE 1.

4. BLACKBODY RADIATION – BB 101

4.1 Brief Synopsis of BB History

Electromagnetic (EM) radiation can be described by four attributes: wavelength, intensity, polarization, and coherence.[95, 96]. In the study of blackbody radiation one seeks a relation between wavelength and intensity in the special case of radiation in steady-state equilibrium inside an isolated enclosure by an ensemble of completely incoherent harmonic oscillators, with random polarizations, at a uniform absolute temperature, T [9–11, 13, 17, 97]. Attempts at an explanation of the form of the intensity versus wavelength curve, from classical modal equipartition and thermodynamic

Table 1: Table of linewidths and Q values for various resonant structures. The Hg linewidth assumes radiation damping only. Other tables are given in Green[73], and in Smith[77].

System	Q	Remarks	Reference
Golf ball	10	$C_R = 85\%$ [94]	[73, 77]
GaN LED	25	16nm linewidth @ 400nm	[98, 99]
BK7 glass	700	0.3 – 2.5 μm band	[93]
Nd:YAG glass laser	$1.3 \cdot 10^{+3}$	210 GHz linewidth @ 1064nm	[99]
Ruby laser	$7.2 \cdot 10^{+3}$	60 GHz linewidth @ 694.3nm	[99]
He-Ne gas laser	$3.2 \cdot 10^{+5}$	1.5 GHz linewidth @ 632.8nm	[99]
Si wineglass resonator	$4 \cdot 10^{+6}$	2 MHz @ RT	[100, 101]
Quartz resonators	$5 \cdot 10^{+6}$	2.5 – 5 MHz @ RT	[102–105]
Quartz resonators	$50 \cdot 10^{+6}$	2.5 – 5 MHz @ 4.1K	[102–105]
Green Hg line	$4.6 \cdot 10^{+8}$	1.2 MHz linewidth @ 546.1nm	[6]
Earth spin-down	$2.6 \cdot 10^{+12}$	$\Delta t \approx 2.3 \text{ ms/cy}$	[106]

arguments led to puzzling and contradictory results. The Rayleigh-Jeans (R-J) equipartition result [22–25], gave good agreement with experiment at long wavelengths, but predicted an unbounded result at short wavelengths; the famous “ultraviolet catastrophe.”[26], At the other extreme, Wien’s thermodynamic result [16–21], was in agreement with experiment at short wavelengths, but failed at longer wavelengths in what might be called the “infrared shortfall.” By interpolating between these limiting cases,⁷ thereby requiring the discretization of energy[27], Planck [28–31] ultimately reached the correct expression, which reduced to the earlier expressions in their correct limits, as well as to the known T^4 variation of total radiated power with absolute temperature, the earlier Stefan-Boltzmann law [12–15]. Figure 2 portrays the situation. The one pre-Planckian experimental truth that survived the quantum revolution was Wien’s Displacement law: The spectral peak occurs at a wavelength (λ_p) inversely proportional to T.[8, 18]; the Stefan-Boltzmann law is a special case of the Displacement law. A much richer account of BB history is given by Richtmyer[5].

4.2 The Spectral Form of BB Radiation

According to the quantum view, the “shape” of any spectrum consists of a steady-state average of an innumerable number of discrete events; this being consonant with the traditional description of BB radiation as a continuous function of frequency (ν) or of wavelength (λ) because of the smallness of Planck’s constant. The exact shape of the blackbody spectrum depends not only on the absolute temperature, but on the dispersion (bookkeeping) rule adopted: wavelength, “intensity (or other related quantity) per unit λ increment,” or frequency, “intensity (or other related quantity) per unit ν increment” parameterization. In either case, we give the result in scaled form, such that the maxima are independent of temperature.

⁷ With S the entropy of an irradiated linear, monochromatic, vibrating resonator, and U the corresponding vibrational energy, the Wien displacement law ($S = f(U/\nu)$) implies that $\partial^2 S / \partial U^2 \propto U^{-1}$, while, for the classical harmonic oscillator, $\partial^2 S / \partial U^2 \propto U^{-2}$. Planck, the master thermodynamicist, interpolated between these by setting $\partial^2 S / \partial U^2 = \alpha / [U(U + \beta)]$, the simplest form yielding agreement with experiment, but requiring introduction of “h”, a new constant of nature. See [30, 33, 34].

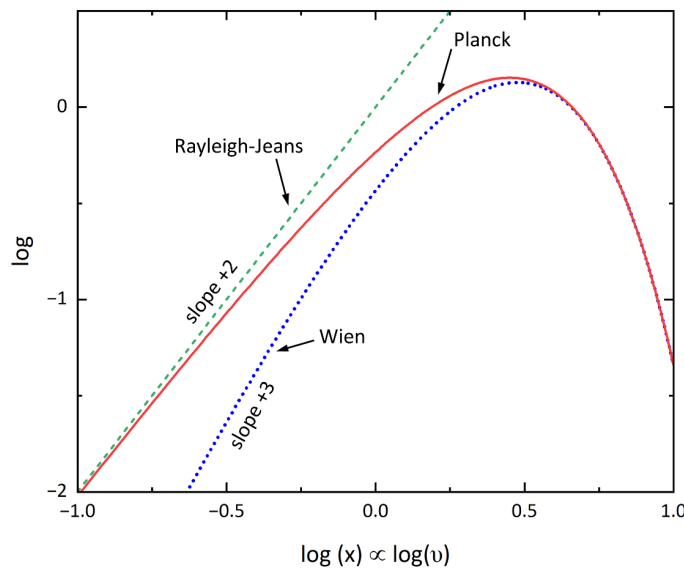


Figure 2: Log-log plots of the Rayleigh-Jeans, Wien, and Planck distribution functions. The Planck function interpolates between the other two, and yields the distribution correct at both frequency limits. The Wien function obeys Maxwell-Boltzmann statistics, whereas the Planck function obeys Bose-Einstein statistics.

Any portrayal of BB spectra involves Boltzmann’s constant k_B , Planck’s constant, h , lightspeed, c , and numerics, and the admixture of these depends on the quantity described. The glossary of names for various quantities associated with BB radiation is oversized; it includes, *inter alia*: spectral radiance, emittance, excitance; monochromatic specific intensity; radiant intensity; spectral energy density; spectral radiance per unit frequency, per unit wavelength; brightness; irradiance; power intensity; etc., etc. The MKS units attached to these terms may agree or not, and one additionally finds in the literature differences in numerical factors assigned to terms bearing the same name. As we use normalized forms these terms are irrelevant for us.

Stripped to essentials, the BB spectra are given by $F_M(X) = X^M/(e^X + n)$, where X , the independent variable, equals $(h\nu/kT) = (hc/\lambda kT) = 1/Y$. M names the parametrization, with $M = 3$ or 5 (ν or λ bookkeeping, respectively); the two M values differ because $\lambda \cdot \nu = c$, so the increments are related by $d\lambda = -c \cdot d\nu/\nu^2$. Coefficient n names the statistics, with $n = -1$ for Planck’s exact result (Bose-Einstein statistics), and $n = 0$ for Wien’s approximation (Maxwell-Boltzmann statistics). Extensions to other M and n values are mentioned briefly in Sec. V. The peak (X_p) of $F_M(X)$ is found from the root of the equation $X \cdot e^X/(e^X + n) = M$, or the alternative form $n \cdot e^{-X} = [(X/M) - 1]$. [9] At the peak, $F_M(X_p) = X_p^M/(e^{X_p} + n)$. For the Planckian cases, $n = -1$, and we set $F_M(X) = Pl_3(\nu) = X^3/(e^X - 1)$ and $Pl_5(\lambda) = X^5/(e^X - 1)$. These are graphed in FIGURE 3 and FIGURE 4, respectively. In each of these scaled forms, the peak is invariant with temperature.

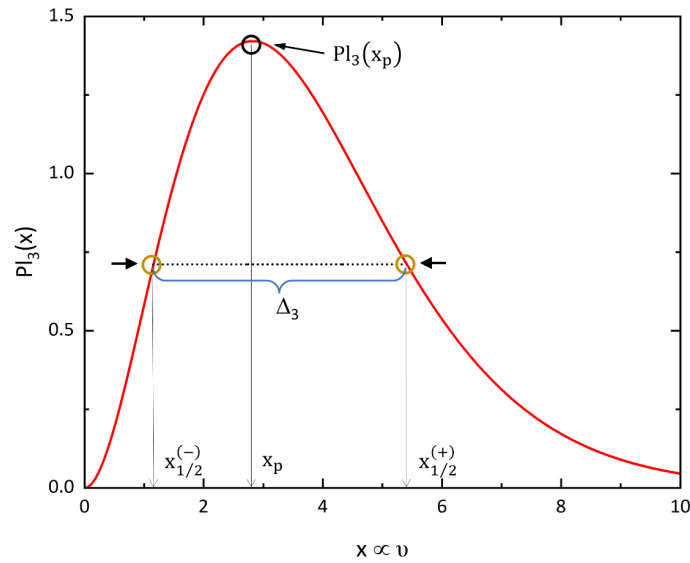


Figure 3: Plot of the normalized quantity $Pl_3(X) = X^3/(e^X - 1)$ vs $X = h\nu/k_B T$, with $1/2$ -power points $X_{1/2}^{(\pm)}$ and $1/2$ -power bandwidth Δ_3 shown. $Pl_3(X) = S_\nu/(T^3 \cdot K_1)$, where $K_1 \propto k_B^3/(c^2 \cdot h^2)$; units of K_1 are $[J/(m^2 \cdot K^3)]$. S_ν is a spectral radiance function.

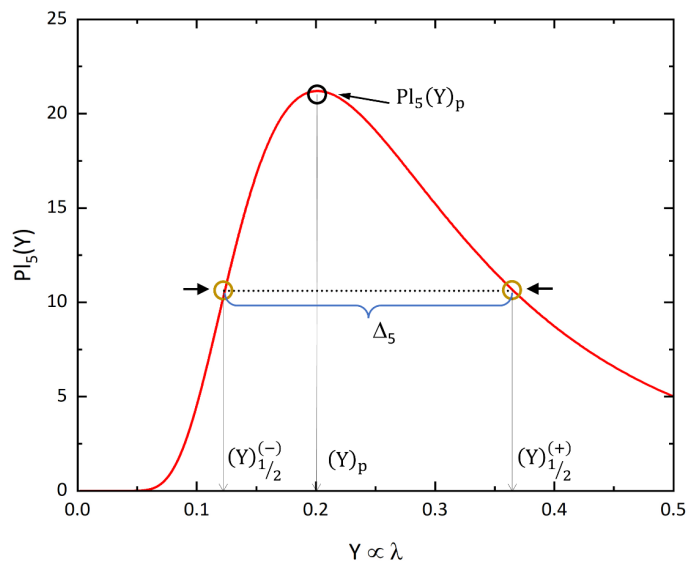


Figure 4: Plot of the normalized quantity $Pl_5(X) = X^5/(e^X - 1)$ vs $Y = 1/X = \lambda k_B T/hc$ with $1/2$ -power points $Y_{1/2}^{(\pm)}$ and $1/2$ -power bandwidth Δ_5 shown. $Pl_5(X) = S_\lambda/(T^5 \cdot K_2)$, where $K_2 \propto k_B^5/(c^3 \cdot h^4)$; units of K_2 are $[J/(m^3 \cdot s \cdot K^5)]$. S_λ is a spectral radiance function.

4.3 Functional Behavior of the BB Spectra at Frequency and Wavelength Limits

Whereas there is an “ultraviolet catastrophe” for the R-J theory, there is no corresponding “infrared catastrophe” for the Wien theory; in the Wien theory the intensity correctly approaches zero in both high and low frequency limits. However, in the Wien theory, the functional variation goes as ν^3 (λ^5) instead of the correct Planckian ν^2 (λ^4) behavior. This is the “IR deficit”; the intensity approaches zero too quickly as ν diminishes to zero. Table 2. shows the various limiting values.

Table 2: Functional variation of the Rayleigh-Jeans, Wien, and Planck BB theories at low and high frequency limits for frequency (ν) and wavelength (λ) dispersion rules.

		$X \rightarrow 0$		$Y = 1/X \rightarrow 0$	
		Rayleigh-Jeans, Planck	Wien	Rayleigh-Jeans	Wien, Planck
ν	X^2	X^3 (IR deficit)	ν	X^2 (UV catastrophe)	$X^3 \cdot e^{-X}$
λ	X^4	X^5 (IR deficit)	λ	X^4 (UV catastrophe)	$X^5 \cdot e^{-X}$

4.4 The Q concept applied to BB spectra

Applying the material discussed in Sec. III to the BB spectra in Fig. 3 and Fig. 4 permits the associated Q values to be found. The effective “Q” of a spectrum, by analogy with ordinary resonance curves such as those in Fig. 1. (b), is taken as $Q = X_p/\Delta$, where Δ is the half-maximum width (“3-dB points”). The half-maximum points are determined from the two roots ($X^{(\pm)}$) of the equation $(\frac{1}{2}X_p^M)/[e^{X_p} - 1] = (X^M)/[e^X - 1]$. The relevant quantities are given, for $M = 3$, in Table 3, and, for $M = 5$, in Table 4. In both cases Q is less than 1, and interestingly, the results are independent of temperature.⁸ One could define blackbody Q in other ways, e.g., $\nu_p/\Delta\nu$ or $\lambda_p/\Delta\lambda$ using either $M = 3$ or 5 values of X, etc. For all such permutations, the effective $Q < 1$. Given, for example in Table 4. are $Q_5(\lambda) = \lambda_p/\Delta\lambda \approx 0.831$ and $Q_5(\nu) = \nu_p/\Delta\nu \approx 0.926$.

Table 3: Pertinent locations on the Planckian distribution function (Pl_3) with frequency parameterization ($M = 3$).

Location	X	$Pl_3(X)$
Lower 1/2-power, $X_{1/2}^{(-)}$	$X_{1/2}^{(-)} = 1.1575$	0.7107
Peak	$X_p = 2.8214$	1.4214
50%-area divisor	$X_{50\%} = 3.5030$	1.3343
Upper 1/2-power, $X_{1/2}^{(+)}$	$X_{1/2}^{(+)} = 5.4116$	0.7107
$\Delta_3 = [X_{1/2}^{(+)} - X_{1/2}^{(-)}] \approx 4.2541$		$Q_3 \approx 0.663$

⁸ The temperature independence holds for the “N-dB” points, not just for the “3 dB” points; moreover, it is also independent of the M value. For the 1/2-power (aka 3 dB) points, $(\frac{1}{2} \cdot X_p^M)/[\exp(X_p) - 1]$ is a constant for a given value of M. The factor 1/2, designating the half-power-points, could be any other value, e.g., the $1/3^{rd}$ (≈ 4.77 dB) or $1/100^{th}$ (20 dB) power-points, etc. The roots become spread further apart as the dB value increases, and the definition of $Q = X_p/\Delta$ must be adjusted accordingly.

Table 4: Pertinent locations on the Planckian distribution function (Pl_5) with wavelength parameterization ($M = 5$).

Location	Y	$Pl_5(Y)$	X
Lower $\frac{1}{2}$ -power	$Y_{1/2}^{(-)} = 0.1235$	10.6007	8.0966
50%-area divisor	$Y_{50\%} = 0.1779$	20.3908	5.6218
Peak	$Y_p = 0.2014$	21.2014	4.9651
Upper $\frac{1}{2}$ -power	$Y_{1/2}^{(+)} = 0.3660$	10.6007	2.7326
$\Delta_5 = [Y_{1/2}^{(+)} - Y_{1/2}^{(-)}] \approx 0.2424$	$Q_5(\lambda) \approx 0.831$	$Q_5(\nu) \approx 0.926$	

5. COMPARISON OF STATISTICS AND M VALUES

Heald[9] considers other cases of integer dispersion rules. In addition to the rules $M = 3$ (ν rule) and $M = 5$ (λ rule), that we have discussed, he mentions $M = 2$ (ν^2 rule), and $M = 4$ (logarithmic rule = intensity per percentage bandwidth, with peak at $X_p \approx 3.9207$); of particular note is his suggestion that the median (50% of energy) rule be considered. The 50% divisor points are given in Table 3. and Table 4. Another criterion of interest is the fraction of area (energy) confined between the half-power points to the total area. Examples are shown in Table 5., along with the associated Q values. If M is considered as a continuous variable, then as M increases, X_p approaches M from below; (for $M = 4$, it is within 2%); as M approaches 1 from above, X_p approaches 0, and the peak value approaches 1 from below. For $M \geq 2$, the fraction of total area between the half-power points is greater than 75%.

Table 5: Fractions of areas between $\frac{1}{2}$ -power points to total area, and Q values associated with various line shapes. The Gaussian and Lorentzian functions describe, e.g., specific types of spectral line broadening in gases and plasmas. Doppler broadening, due to thermal motion, is represented by the Gaussian function. The Lorentzian function is used in connection with natural broadening (finite radiative lifetimes), and collisional / pressure broadening (finite lifetimes due to collisions).^{9, 10} The first three entries are symmetrical in X; the others are not. The Gaussian ratio equals $\text{erf}[\ln(2)]$.

Spectral line shape, normalized	Ratio (%)	Q	Comments
Gaussian	$\exp(-(\ln(2) \cdot x^2))$	≈ 76.10	$\frac{1}{2}$ $\frac{1}{2}$ -power points at $X = \pm 1$
Lorentzian	$1/(1 + X^2)$	50	$\frac{1}{2}$ $\frac{1}{2}$ -power points at $X = \pm 1$
RLC/BVD	$1/[1 + Q^2 \cdot (X - 1/X)^2]$	50	Q variable parameter Q
Bose-Einstein (Planck)	$X^3/(e^X - 1)$	≈ 75.36	0.663 ν dispersion
Bose-Einstein (Planck)	$X^5/(e^X - 1)$	≈ 75.36	0.831 λ dispersion
Maxwell-Boltzmann (Wien)	$X^3/(e^X - 0)$	≈ 74.81	0.726 ν dispersion
Fermi-Dirac	$X^3/(e^X + 1)$	≈ 74.46	0.768 ν dispersion

⁹ These functions may be combined, on the assumption that both mechanisms are independent, yielding the Voigt profile as their convolution. The results must be obtained by numerical integration. The 50%-area point depends, moreover, on the admixture of the distributions. See Desmond Walter Posener, "The shape of spectral lines: Tables of the Voigt profile," Australian Journal of Physics **12**(2) (1959) 184-196; Vivek Bakshi and Robert J. Kearney, "New tables of the Voigt function," Journal of Quantitative Spectroscopy and Radiative Transfer **42**(2) (1989) 111-115.

¹⁰ The index n may also be a rational fraction; this is a feature of quasiparticles in 2-D structures; See Jon Magne Leinaas and Jan Myrheim, "On the theory of identical particles," Il Nuovo Cimento **37B**(1) (1977) 1-23; James Nakamura, Shuang Liang, Geoffrey

Table 6: Q values computed by the 1/2-power method for three distributions, assuming frequency parameterization ($M = 3$). The “resonance” curve is $X^3/(e^X + n)$ versus X . For $X \rightarrow 0$, the R-J and the asymptotic Planck slopes equal +2 on the logarithmic graph of Fig. 2., while the Wien asymptotic slope is +3. In this limit, any graph with $n > 0$ has an asymptotic slope of +3, but falls below the Wien curve; for example, the F-D curve has an ordinate $[3 \cdot \log(X) - \log(2)]$.

Statistics	$X^3/(e^X + n)$	$X_{1/2}^{(-)}$	X_p	$X_{1/2}^{(+)}$	Q
Bose-Einstein	$n = -1$	1.157465	2.821439	5.411575	0.6632
Maxwell-Boltzmann	$n = 0$	1.394137	3.	5.525350	0.7262
Fermi-Dirac	$n = +1$	1.536495	3.131020	5.616138	0.7675

6. CONCLUSION

Ascribing the EE resonance parameter “quality factor” to the graphs arising from the physics concept of BB radiation might seem more than a bit incongruous, but in the spirit of paying tribute to Kirchhoff’s bicentennial, we have done just that! Apart from its obvious didactic value, the exercise reveals that the equivalent “Q” of this totally incoherent photon fluid is lower than unity, (as might be anticipated). However, this Q measure unexpectedly turns out to be independent of both the temperature, and of the dispersion rule adopted.

AUTHOR DECLARATIONS

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References

- [1] Kirchhoff GR. IV. Ueber das Verhältniß zwischen dem Emissionsvermögen und dem Absorptionsvermögen der Körper für Wärme und Licht. *Ann Phys.* 1860;185:275-301. Available at: <https://onlinelibrary.wiley.com/toc/15213889/1860/185/2>
- [2] Kirchhoff S. I. Ueber den Durchgang eines elektrischen Stromes durch eine Ebene, insbesondere durch eine kreisförmig. *Ann Phys.* 1845;140:497-514. ^[F1,F2] Available at: <https://onlinelibrary.wiley.com/toc/15213889/1845/140/4>
- [3] Wien W (W), Lummer O. Methode zur Prüfung des Strahlungsgesetzes absolut Schwarzer Körper. *Ann Phys.* 1895;292:451-456. Available at: <https://onlinelibrary.wiley.com/toc/15213889/1895/292/11>

C. Gardner, and Michael J. Manfra, “Fabry-Pérot interferometry at the $\nu = 2/5$ fractional quantum Hall state,” *Physical Review X* **13**(4) (2023) 041012.

- [4] Wien W (W). Ueber die Energievertheilung im Emissionsspectrum eines Schwarzen Körpers. Ann Phys. 1896;294:662-669. Available at: <https://onlinelibrary.wiley.com/toc/15213889/1896/294/8>
- [5] Richtmyer FK. Introduction to Modern Physics. 1st ed. New York & London: McGraw-Hill Book Company; 1928:177-247. In All of the many subsequent editions the material devoted to blackbody phenomena becomes progressively shorter.
- [6] Von Hippel AR. Dielectrics and Waves. 1st ed. 2nd ed. London: Chapman & Hall. Boston. Artech House 1995:106-108.
- [7] Klein MJ. Thermodynamics and Quanta in Planck's Work. Phys Today. 1966;19:23-32.
- [8] Boyer TH. Thermodynamics of the Harmonic Oscillator: Wien's Displacement Law and the Planck Spectrum. Am J Phys. 2003;71:866-870.
- [9] Heald MA. Where Is the 'Wien Peak'? Am J Phys. 2003;71:1322-1323.
- [10] Gearhart C. Black-Body Radiation. In: Greenberger D, Hentschel K, Weinert F, editors. Compendium of Quantum Physics. Berlin, Heidelberg: Springer; 2009:39-42.
- [11] Marr JM, Wilkin FP. A Better Presentation of Planck's Radiation Law. Am J Phys. 2012;80:399-405.
- [12] Stefan J. Über die Beziehung zwischen der warmestrahlung und der temperatur. Sitzungsber Kaiserlichen Akad Wissenschaften Math-Naturwiss Cl Abt. 1879;2:391-428. Available here: <https://archive.org/details/sitzungsbericht543klasgoog/page/390/mode/2up>
- [13] Boltzmann LE. Ueber eine von Hr. Bartoli entdeckte Beziehung der Wärmestrahlung zum Zweiten Hauptsatze. Ann Phys. 1884;258:31-39. Available here: <https://onlinelibrary.wiley.com/toc/15213889/1884/258/5>
- [14] Boltzmann LE. Ableitung des Stefan'schen Gesetzes, betreffend die Abhängigkeit der Wärmestrahlung von der Temperatur aus der electromagnetischen Lichttheorie. Ann Phys. 1884;258:291-294. Available here: <https://onlinelibrary.wiley.com/toc/15213889/1884/258/6>
- [15] Crepeau J. A Brief History of the T4 Radiation Law. Proceedings of ASME 2009 Heat Transfer Summer Conference, San Francisco; 2009. Available from: paper HT2009-88060.
- [16] Wien W (W). Ueber den Begriff der Localisirung der Energie. Ann Phys. 1892;281:685-728. Available here: <https://onlinelibrary.wiley.com/toc/15213889/1892/281/4>
- [17] Wien W (W). Die obere Grenze der Wellenlängen, welche in der Wärmestrahlung fester Körper Vorkommen können; Folgerungen aus dem Zweiten Hauptsatz der Wärmetheorie. Ann Phys. 1893;285:633-641. Available at: <https://onlinelibrary.wiley.com/toc/15213889/1893/285/8>
- [18] Wien W (W). Eine neue Beziehung der Strahlung Schwarzer Körper zum Zweiten Hauptsatz der Wärmetheorie. Sitzungsber K Preußischen Akad Wissenschaften Berl Erster Halb-band. 1893:55-62. Available at: <https://digilib.bbaw.de/digilib/digilib.html?fn=/silo10/Bibliothek.tiff/10-sitz/1893-1/tif&pn=2>

- [19] Wien W (W). Temperatur und Entropie der Strahlung. *Ann Phys.* 1894;288:132-165. Available here: <https://onlinelibrary.wiley.com/toc/15213889/1894/288/5>
- [20] Wien W (W). Ueber die Energievertheilung im Emissionsspectrum eines Schwarzen Körpers. *Ann Phys.* 1896;294:662-669. Available here: <https://onlinelibrary.wiley.com/toc/15213889/1896/294/8>
- [21] Wien W. XXX. On the Division of Energy in the Emission-Spectrum of a Black Body. *Philos Mag (London)*. 1897;43:214-220.
- [22] Lord Rayleigh L. LIII. Remarks Upon the Law of Complete Radiation. *Philos Mag (London)*. 1900;49:539-540.
- [23] Lord Rayleigh . The Dynamical Theory of Gases and of Radiation. *Nature*. 1905;72:54-55.
- [24] Jeans JH. XI. On the Partition of Energy Between Matter and ÆTher. *Philos Mag (London)*. 1905;10:91-98.
- [25] Lord Rayleigh . The Constant of Radiation as Calculated From Molecular Data. *Nature*. 1905;72:243-244.
- [26] Ehrenfest P. Welche Züge der Lichtquantenhypothese Spielen in der Theorie der Wärmestrahlung eine Wesentliche Rolle? *Ann Phys.* 1911;341:91-118. Available at: <https://onlinelibrary.wiley.com/toc/15213889/1911/341/11>
- [27] Boltzmann LE. Über die Beziehung dem Zweiten Hauptsatze der mechanischen Wärmetheorie und der Wahrscheinlichkeitsrechnung respektive den Sätzen über das Wärmegleichgewicht (Complexionen-Theorie). *Sitzungsberichte der Kaiserlichen Akademie der Wissenschaften. mathematisch-naturwissenschaftliche classe*, 76 1877 373-435; Reprinted in *Wissenschaftliche Abhandlungen von Ludwig boltzmann*. Vol. 2. New York: Chelsea Publishing Company; 1968:164-223. Available here: <https://archive.org/details/wissenschaftlich0000bolt/page/n5/mode/2up>
- [28] Planck M. Über eine Verbesserung der Wien'schen Spectralgleichung. *Verh Dtsch Phys Ges.* 1900;2:202-204. Available at: <https://hdl.handle.net/2027/coo.31924056107224>
- [29] Planck M. Entropie und Temperatur strahlender Wärme. *Ann Phys.* 1900;306:719-737. Available at: <https://onlinelibrary.wiley.com/toc/15213889/1900/306/4>
- [30] Planck M. Zur theorie des Gesetzes der Energieverteilung im normalspectrum. *Verh Dtsch Phys Ges.* 1900;2:237-245. Available at: <https://hdl.handle.net/2027/coo.31924056107224>
- [31] Planck M. Uber das Gesetz der Energieverteilung in Normalspectrum. *Ann Phys.* 1901;309:553-563. Available at: <https://onlinelibrary.wiley.com/toc/15213889/1901/309/3>
- [32] Epstein PS. On Planck's Quantum of Action. *Am J Phys.* 1954;22:402-405.
- [33] Straneo P, "Omogeneità della equazioni e Similitudine nella fisica," *Atti della Accademia nazionale dei Lincei* 1917 ;26:271-276. Available at: http://periodici.librari.beniculturali.it/visualizzatore.aspx?anno=1917&ID_testata=30&ID_periodico=9073

- [34] Available here: <https://arxiv.org/abs/cond-mat/0101293>.
- [35] Nauenberg M. Max Planck and the Birth of the Quantum Hypothesis. *Am J Phys.* 2016;84:709-720.
- [36] Volta A. XVII. On the Electricity Excited by the Mere Contact of Conducting Substances of Different Kinds. *Philosophical transactions of the Royal Society (London)*. 1800;90:403-431.
- [37] Ørsted HC. Experiments on the Effect of a Current of Electricity on the Magnetic Needle. *Annals Philos (Lond)*. 1820;16:273-276.
- [38] Ampère AM. Théorie des phénomènes électro-dynamiques, uniquement déduite de l'expérience. 1st ed Godfrey MD, translator. Paris: Méquignon-Marvis; English translation: *Mathematical theory of electrodynamic phenomena, uniquely derived from experiments*. Stanford University; 2021. Available at: <https://books.google.com/books?id=LzcVAAAAQAAJ>
- [39] Ampère AM. Mémoire sur la théorie mathématique des phénomènes électro-dynamiques uniquement déduite de l'expérience. *Mémoires de l'Académie Royale des Sciences de l'institut de France* 1823;6:175-387. Available at: <https://gallica.bnf.fr/ark:/12148/bpt6k3221x/f363.item>
- [40] Available here: <https://books.google.com/books?id=wBtkAAAAcAAJ>.
- [41] Ohm GS. *Die galvanische Kette, mathematisch bearbeitet*. Berlin: T H Riemann; 1827. Available at: <https://books.google.com/books?id=wBtkAAAAcAAJ>
- [42] Faraday M. *Experimental Researches in Electricity*. Vol. 1 (London, Richard & John Edward Taylor, 1839).
- [43] Faraday M. *Experimental Researches in Electricity*. Vol. 2 (London, Richard & John Edward Taylor, 1844).
- [44] Faraday M. *Experimental Researches in Electricity*. Vol. 3 (London, Richard & John Edward Taylor, 1855).
- [45] Maxwell JC. A Dynamical Theory of the Electromagnetic Field. *Philos Trans R Soc Lond (London)*. 1865;155:459-512.
- [46] Maxwell JC. *A Treatise on Electricity and Magnetism*. 3rd ed. London: Oxford University Press; New York: Dover; 1954.
- [47] Ballato A. Modeling Piezoelectric and Piezomagnetic Devices and Structures via Equivalent Networks. *IEEE Trans Ultrason Ferroelectr Freq Control*. 2001;48:1189-1240.
- [48] Hansen WW. A Type of Electrical Resonator. *J Appl Phys*. 1938;9:654-663.
- [49] Dicke RH. General Microwave Circuit Theorems. In: *Principles of Microwave Circuits of MIT Radiation Laboratory Series*. 1948; 8.130-161.
- [50] Fano RM, Chu LJ, Adler RB. *Electromagnetic Fields, Energy, and Forces*. Cambridge, MA: M.I.T. Press; 1960.

- [51] Adler RB, Chu LJ, Fano RM. Electromagnetic Energy Transmission and Radiation. New York: John Wiley & Sons; 1960.
- [52] Montgomery CG, Dicke RH, Purcell EM. Principles of Microwave Circuits of MIT Radiation Laboratory Series. New York: McGraw-Hill; 1948.
- [53] Beringer R. Resonant Cavities as Microwave Circuit Elements. In principles of Microwave Circuits. New York: McGraw-Hill. 1948;8:209-237.
- [54] Available from: <http://ocw.mit.edu>.
- [55] Leiden Jars Were Re-Named 'Condensers,' Becoming Finally 'Capacitors.' Similarly, "Coils" Became "Inductors."
- [56] Heaviside O. Electrical Papers. Vol. 1. London and New York: Macmillan Company; 1892.
- [57] Heaviside O. Electrical Papers. Vol. 2. London and New York: Macmillan Company; 1894.
- [58] Heaviside O. Electromagnetic Theory. Vol. 1 (London. The electrician. Print Publ Co. 1893).
- [59] Heaviside O. Electromagnetic Theory. Vol. 2 (London. The electrician. Print Publ Co. 1899).
- [60] Heaviside O. Electromagnetic Theory. Vol. 3 (London. The electrician. Print Publ Co. 1912).
- [61] Jeffreys H. Operational Methods in Mathematical Physics. Cambridge: Cambridge University Press; 1927.
- [62] Robertson BL. Operational Method of Circuit Analysis. Trans Am Inst Electr Eng. 1935;54:1037-1045.
- [63] Fich S. Transient Analysis in Electrical Engineering. New York: Prentice Hall; 1951.
- [64] Guillemin EA. Introductory circuit theory. New York: John Wiley & Sons. 1953.
- [65] Guillemin EA. Synthesis of Passive Networks: Theory and Methods Appropriate to the Realization and Approximation Problems. New York: John Wiley & Sons; 1957.
- [66] Belevitch V. Summary of the History of Circuit Theory. Proc IRE. 1962;50:848-855.
- [67] Carlin HJ, Giordano AB. Network Theory: An Introduction to Reciprocal and Nonreciprocal Circuits. Englewood Cliffs, NJ: Prentice Hall; 1964.
- [68] Darlington S. A History of Network Synthesis and Filter Theory for Circuits Composed of Resistors, Inductors, and Capacitors. IEEE Trans Circuits Syst. 1984;31:3-13; reprinted: 1999;461:4-13.
- [69] Maloberti F, Davies AC. A Short History of Circuits and Systems. Aalborg, Denmark: River Publishers; 2016.
- [70] White RM, Doering RW. Electrical Engineering Uncovered. 2nd ed. Upper Saddle River, NJ: Prentice Hall; 2001.
- [71] Johnson KS. Transmission Circuits for Telephonic Communication. New York: D. Van Nostrand Co.; 1924.

- [72] Legg VE, Given FJ. Compressed Powdered Molybdenum Permalloy for High Quality Inductance Coils. *Bell Syst Tech J.* 1940;19:385-406. The term “Coil quality factor, Q” appears on page 394.
- [73] Green EI. The Story of Q. *Am Sci.* 1955;43:584-594.
- [74] Jeffreys B. A Q-Rious Tale; The Origin of the Parameter Q in Electromagnetism. *Q J R Astron Soc.* 1985;26:51-52.
- [75] McMullan D. A Note on the Origin of the Parameter Q in Electromagnetism. *Q J R Astron Soc.* 1985;26:529.
- [76] Jeffreys B. A Question About 'Q'. *Electron Wirel World.* 1598;92:17.
- [77] Smith KL, “Q,” *Electronics & Wireless World.* 1985;93:51-53; volume number is misprinted as 92.
- [78] McMullan D. A Question About Q. *Electron Wirel World.* 1601;93:62.
- [79] Fellgett PB. Quality Factor 'Q'. *J Inst Electron Radio Eng UK.* 1985;55:276.
- [80] Fellgett PB. Q-Riouser and Q-Riouser. *Q J R Astron Soc.* 1986;27:300-301.
- [81] Jeffreys B. A Footnote to 'A Q-Rious Tale'. *Q J R Astron Soc.* 1986;27:693-694.
- [82] Smith KL. On the Origins of 'Quality Factor Q'. *Q J R Astron Soc.* 1986;27:695-696.
- [83] Jeffreys B. An Answer About 'Q'. *Electron Wirel World.* 1602;93:37.
- [84] Fellgett PB. Origins of the Usage of Q. *J Inst Electron Radio Eng (IERE) Lond.* 1986;56:45-46.
- [85] Fellgett PB. Correction: Origins of the Usage of 'Q'. *J Inst Electron Radio Eng UK.* 1986;56:297.
- [86] Montgomery CG Chapter 6. Principles of Microwave Circuits. In: *Waveguide Circuit Elements of MIT Radiation Laboratory Series.* 1948; 8:162-206:Article 6.14 182-186 gives the relations $Q \propto \omega \cdot \partial B / \partial \omega$ and $Q = \omega \cdot \text{Energy stored} / [\text{Energy dissipated/sec}]$
- [87] Beringer R Chapter 7. Principles of Microwave Circuits. In: *Resonant Cavities as Microwave Circuit Elements of MIT Radiation Laboratory Series.* New York: McGraw-Hill; 1948;8:207-239:Article 7.9 230-231 gives the relations $Q = 2\pi \cdot \text{Energy stored} / [\text{Energy dissipated/cycle}]$ and $Q \propto \omega \cdot \partial B / \partial \omega = \omega \cdot \partial X / \partial \omega$.
- [88] Ballato A. Resonance in Piezoelectric Vibrators. *Proc IEEE.* 1970;58:149-151.
- [89] Feld DA, Parker R, Ruby R, Bradley P, Dong S. After 60 Years: A New Formula for Computing Quality Factor Is Warranted. In: *Ieee Ultrasonics Symposium Proceedings;*2008:431-436.
- [90] Ohira T. What in the World Is Q? *IEEE Microw Mag.* 2016;17:42-49.
- [91] Ohira T. The Kq Product as Viewed by an Analog Circuit Engineer. *IEEE Circuits Syst Mag.* 2017;17:27-32.

- [92] Montgomery CG . Principles of Microwave Circuits. In: Elements of Network Theory of MIT Radiation Laboratory Series. New York: McGraw-Hill; 1948;8:127-129:Article 4.13 gives the complex input impedance for a series RLC circuit as $Z/R = [1 + j \cdot Q \cdot (\Omega - 1/\Omega)]$.
- [93] Ballato A, Ballato J. Sellmeier Circuits: A Unifying View on Optical and Plasma Dispersion Fitting Formulas. J Appl Phys. 2020;128:034901.
- [94] The Rebound of an Elastic Ball on a Rigid Surface Is Characterized by a Coefficient of Restitution (C_R) Equal To $\sqrt{h/H}$, Where H = Drop Height and H = Height of Rebound. In Terms of C_R the Q is $(-\pi/2)/\ln(C_R)$.
- [95] Born M, Wolf E. Principles of Optics. 7th ed. Cambridge: Cambridge University Press;1999.
- [96] Ballato J, Gupta MC. The Handbook of Photonics. 2nd ed. Boca Raton: CRC Press; 2006.
- [97] Stokes GG. On the Composition and Resolution of Streams of Light From Different Sources. Trans Camb Philos Soc. 1852;9:399-416.
- [98] Saleh BEA, Teich MC. Fundamentals of Photonics. New York: John Wiley & Sons; 1991.
- [99] Tran MA, Huang D, Bowers JE. Tutorial on Narrow Linewidth Tunable Semiconductor Lasers Using Si/III-v Heterogeneous Integration. Appl Phys Lett Photonics. 2019;4:111101.
- [100] Lee JEY, Seshia AA. Square Wine Glass Mode Resonator With Quality Factor of 4 Million Seventh Ieee Sensors Conference, Lecce, Italy; 2008:1257-1260.
- [101] Glass Wine Goblet, Q Not Measured: It Appears That Authors Are More Inclined to Shatter Wine Glasses by Acoustic Radiation Than to Measure Their Acoustic Q Values Prior to the Denouement.
- [102] Bömmel HE, Mason WP, Warner AW, Jr. Experimental Evidence for Dislocations in Crystalline Quartz. Phys Rev. 1955;99:1894-1896.
- [103] Bömmel HE, Mason WP, Warner AW, Jr. Dislocations, Relaxations, and Anelasticity of Crystal Quartz. Phys Rev. 1956;102:64-71.
- [104] Warner AW, Jr. Ultra-Precise Quartz Crystal Frequency Standards. Inst Radio Eng (IRE) Trans Instrum. 1958;7:185-188.
- [105] Warner AW, Jr. Design and Performance of Ultraprecise 2.5-Mc Quartz Crystal Units. Bell Syst Tech J. 1960;39:1193-1217.
- [106] Stephenson FR. Historical Eclipses and Earth's Rotation. Astron Geophys. 2003;44:2.22-2.27.