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Generalized Hertz Vector in the Dissipative Electrodynamics

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Abstract

In the electromagnetic theory, the Hertz vector reduces the number of potentials in the free fields. The further advantage of this potential is that it is much easier to solve particular radiation processes. It indicates that the related method is sometimes more effective than the scalar and vector potential-based relations. Finally, the measurable field variables, the electric and magnetic fields, can be deduced by direct calculation from the Hertz vector. However, right now, the introduction of the Hertz vector operates if the conductive current density $\mathbf{j} = \sigma \mathbf{E}$ is neglected. We suggest a generalization by the conductive currents, i.e., when the electromagnetic field dissipates irreversibly to Joule heat. The presented procedure enables us to introduce also the Lagrangian formulation of the discussed dissipated electromagnetic waves. It paves a new way involving damping physical fields in a thermodynamic frame.

Keywords: Dissipative electrodynamics, Generalized Hertz vector, Irreversible loss, Lagrangian formulation, Telegrapher's equation

1. MOTIVATION

The material conductivity strongly effects the electromagnetic wave propagation which interaction may cause an energy loss but it may allow us to get information about the property of matter. The microwave cavity perturbation experiments [1], are generally used to explore the material structure measuring the resonance frequency [2]. Wide range of electric conductive novel materials can be studied like Li_4C_{60} superionic conductor [3], single wall carbon nanotubes [4], with these high accuracy measurements [5]. It seems that this experimental setup is applicable to study superconductor

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powders [6]. We can conclude from the results that the heating effects can be also in the focus of the observations. The electromagnetic wave and thermodynamic heat couping appears in the so-called coupled electromagnetic and thermal skin effect problems [7], which is a boundary layer approximation. This effect may play an important role in low dimensional phenomena. Similarly, the surface impedance problems [] may give unexpected physical effects in the electrodynamic and thermal interactions with the quantum property of materials. Hopefully, the presented generalization of the Hertz vector can be applied to study such physics as superconducting films in the surface impedance measurements [10].

Based on these physical motivations our aim is to formulate the electrodynamics of the conductive materials, treating by a single potential function. This potential can help us to couple the different fields like electric and thermal conductions. Hertz introduced this vector field by which he deduced the free electromagnetic field. Here, we show that the generalized Hertz vector field makes allow the description of energy dissipation related to the conductive currents. This generalization is the first step in the formulation by which the electrodynamic field may be coupled with the other fields in the future.

2. GENERALIZED VECTOR POTENTIAL, HERTZ VECTOR AND MODIFIED LORENZ CONDITION

Certain mathematical operators, such as the first-time derivatives, can bring irreversible and dissipative behavior in theory, as in the case of thermal conduction and the damped oscillator [11–14]. This minor change in the structure of equations causes a particular challenge in the construction of Lagrangian [15–18]. Now, are going to show you that the Joule heat brings another twist to the Lagrangian formulation of the electromagnetic theory.

In the case of free electrodynamic fields, the introduction of vector potential is essential to deduce from Lagrangian formulation since the Maxwell equations contain non-selfadjoint operators (first-time derivative and divergence). The vector and scalar potentials generate the measurable physical variables. Mathematically these potentials are similarly applied, such as in the previous dissipative processes [11–13]. In general, we can recognize that the introduction of the potentials is independent of dissipation. (For a detailed comparative study of the requirements, please consult with Ref. [19].)

Originally, Faraday introduced the term $-\dot{\mathbf{A}}$ as an electrostatic force relating to the so-called "electrotonic state" [20, 21]. Fortunately, the usage of the scalar and vector potential is also possible in the case of the existence of charges and currents [22, 23].

This fact shows that we can deduce the free electromagnetic field from a single generator field [24, 25]. There are electromagnetic problems that cannot be solved or are complicated to elaborate without the Hertz vector [26]. The symmetry of the Maxwell equations allow an alternative introduction of the Hertz vector involving its gauge symmetries [27–29]. Ornigotti *et al*. showed that due to the transversality of the electromagnetic wave, the Hertz vector can be expressed as a product of a constant polarization vector and a scalar potential [30]. It may give the next physical conception to the Hertz vector formulations.

We aim to achieve the Hamiltonian structure of Joule heat-caused dissipative electrodynamics. In the presence of electrically conductive materials the Maxwell equations are

$$\frac{1}{\mu_0}\mathrm{curl}\mathbf{B} = \sigma\mathbf{E} + \varepsilon_0\dot{\mathbf{E}}, \tag{1a}$$

$$\mathrm{curl}\mathbf{E} = -\dot{\mathbf{B}}, \tag{1b}$$

$$\mathrm{div}\mathbf{E} = 0, \tag{1c}$$

$$\mathrm{div}\mathbf{B} = 0. \tag{1d}$$

The $\mathbf{j} = \sigma\mathbf{E}$ term leads to the Joule heat generation. From a thermodynamical viewpoint, its importance is unquestionable. The difficulty is how to generalize the above formalism for the present case [25]. It can be seen that the new term, $\sigma\mathbf{E}$, is troublesome difficulties. Let us introduce a generalized definition of the vector potential $\mathbf{A}_m$ and the Hertz vector, $\mathbf{\Pi}_m$ involving the term related to the current

$$\begin{aligned} \mathbf{A}_m &= \varepsilon_0\mu_0\dot{\mathbf{\Pi}}_m + \sigma\mu_0\mathbf{\Pi}_m, \\ \varphi &= -\mathrm{div}\mathbf{\Pi}_m, \end{aligned} \tag{2}$$

and the modified Lorenz condition

$$\mathrm{div}\mathbf{A}_m + \varepsilon_0\mu_0\dot{\varphi} + \sigma\mu_0\varphi = 0. \tag{3}$$

Here, we can point out the role of the Hertz vector $\mathbf{\Pi}_m$. The formulated field quantities $\mathbf{E}$ and $\mathbf{B}$ are

$$\begin{aligned} \mathbf{E} &= -\varepsilon_0\mu_0\ddot{\mathbf{\Pi}}_m - \sigma\mu_0\dot{\mathbf{\Pi}}_m + \mathrm{grad}\,\mathrm{div}\mathbf{\Pi}_m, \\ \mathbf{B} &= \varepsilon_0\mu_0\mathrm{curl}\dot{\mathbf{\Pi}}_m + \sigma\mu_0\mathrm{curl}\mathbf{\Pi}_m. \end{aligned} \tag{4}$$

We can conclude that a single generator space is still sufficient to produce gauge spaces. The connections between the measurable field quantities $\mathbf{E}$, and $\mathbf{B}$ and the potentials $\mathbf{A}_m$ and φ remain to have the same physical meaning

$$\begin{aligned} \mathbf{E} &= -\dot{\mathbf{A}}_m - \mathrm{grad}\varphi, \\ \mathbf{B} &= \mathrm{curl}\mathbf{A}_m. \end{aligned} \tag{5}$$

One can prove that all of the field quantities complete the requirement of damping wave (telegrapher) equation

$$0 = \varepsilon_0\mu_0\ddot{\mathbf{G}} + \sigma\mu_0\dot{\mathbf{G}} - \nabla^2\mathbf{G}, \tag{6}$$

where $\mathbf{G}$ can be $\mathbf{E}$, $\mathbf{B}$, $\mathbf{A}_m$, φ, and $\mathbf{\Pi}_m$, without any restriction. By the first equation of Eq. (4), the electronic field $\mathbf{E}$ can be expressed in an alternative form

$$\mathbf{E} = \mathrm{curl}\mathrm{curl}\mathbf{\Pi}_m. \tag{7}$$

3. THE LAGRANGIAN FORMULATION

We need to find a suitable Lagrange density function if we would like to deduce the field equation and exploit the Hamilton's principle. The construction is not self-explanatory. However, if it is possible, the existence of Lagrangian is of fundamental importance. Now, the formulated Lagrangian is

$$L = \frac{1}{2}\varepsilon_0 \left(-\dot{\mathbf{A}}_m - \mathrm{grad}\varphi\right)^2 - \frac{1}{2\mu_0}\left(\mathrm{curl}\mathbf{A}_m\right)^2 + \sigma \mathrm{curl}\mathbf{\Pi}_m \mathrm{curl}\mathbf{A}_m$$
$$- \frac{1}{2}\sigma^2\mu_0 \left(\mathrm{curl}\mathbf{\Pi}_m\right)^2 - \frac{1}{2}\varepsilon_0 \mathbf{\Pi}_m \mathrm{curlcurl}\nabla^2\mathbf{\Pi}_m - \frac{1}{2}\varepsilon_0^2\mu_0\left(\mathrm{curl}\dot{\mathbf{\Pi}}_m\right)^2, \quad (8)$$

which pertains to a dissipative process in electrodynamics. The elaboration of variation is necessary for each field function as variables $\mathbf{A}_m$, φ, and $\mathbf{\Pi}_m$. The exactness of Lagrangian is complete if we obtain the relevant equations of motion, or field equations.

Applying the mathematical rules of variational calculus, the variation with respect to the variable $\mathbf{A}_m$ results the first Maxwell equation (1a)

$$0 = -\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{\mathbf{A}}_m} + \mathrm{curl}\frac{\partial L}{\partial \mathrm{curl}\mathbf{A}_m}$$
$$= \varepsilon_0 \frac{\partial}{\partial t}\left(-\dot{\mathbf{A}}_m - \mathrm{grad}\varphi\right) - \frac{1}{\mu_0}\mathrm{curlcurl}\mathbf{A}_m + \sigma\mathrm{curlcurl}\mathbf{\Pi}_m$$
$$0 = \varepsilon_0\dot{\mathbf{E}} - \frac{1}{\mu_0}\mathrm{curl}\mathbf{B} + \sigma\mathbf{E}. \quad (9)$$

We arrive at the third Maxwell equation (1c) by the variation φ

$$0 = -\mathrm{div}\frac{\partial L}{\partial \mathrm{grad}\varphi} = \varepsilon_0 \mathrm{div}\left(-\dot{\mathbf{A}}_m - \mathrm{grad}\varphi\right) = \varepsilon_0\mathrm{div}\mathbf{E}. \quad (10)$$

The variation with respect to $\mathbf{\Pi}_m$ results the Euler-Lagrange equation

$$0 = \frac{\partial L}{\partial \mathbf{\Pi}_m} + \mathrm{curl}\frac{\partial L}{\partial \mathrm{curl}\mathbf{\Pi}_m} + \mathrm{curlcurl}\nabla^2 \frac{\partial L}{\partial \mathrm{curlcurl}\nabla^2\mathbf{\Pi}_m} - \mathrm{curl}\frac{\partial}{\partial t}\frac{\partial L}{\partial \mathrm{curl}\dot{\mathbf{\Pi}}_m}$$

$$= -\varepsilon_0\mathrm{curlcurl}\nabla^2\mathbf{\Pi}_m + \sigma\mathrm{curlcurl}\mathbf{A}_m - \sigma^2\mu_0\mathrm{curlcurl}\mathbf{\Pi}_m + \varepsilon_0^2\mu_0\mathrm{curlcurl}\ddot{\mathbf{\Pi}}_m. \quad (11)$$

Applying the definition of generalized vector potential $\mathbf{A}_m$ in Eq. (2), and after the simplifications, an expected telegrapher's equation appears for the generalized Hertz vector $\mathbf{\Pi}_m$:

$$0 = \varepsilon_0\mu_0\ddot{\mathbf{\Pi}}_m + \sigma\mu_0\dot{\mathbf{\Pi}}_m - \nabla^2\mathbf{\Pi}_m. \quad (12)$$

We can conclude that the Lagrangian in Eq. (8) involves and describes correctly the electric conductivity related term. Due to the Ohmic resistance Joule heat dissipation appears in the considered system. As a conclusion, we may say that this irreversible behavior of the electromagnetic theory is within the frame of the Hamilton's principle.

4. THE ELECTROMAGNETIC ENERGY LOSS

As the Joule dissipation is involved in the Lagrangian, so it needs to contribute to the exposition of the Hamiltonian. This formulation of the loss of electromagnetic field transforming into the thermal field opens a novel way to couple these classical phenomena. The building up the Hamiltonian have the usual mathematical steps.

The canonical momenta exist to those field quantities that stand as pure time derivatives in the Lagrangian. So, in the present case, we can define the canonical momentum just for $\mathbf{A}_m$, as is general in the Hamiltonian formulation

$$P_{\mathbf{A}_m} = \frac{\partial L}{\partial \dot{\mathbf{A}}_m} = -\varepsilon_0 \left(-\dot{\mathbf{A}}_m - \mathrm{grad}\varphi\right). \tag{13}$$

Thus the Hamiltonian of the dissipative electromagnetic field is

$$\begin{aligned} H &= P_{\mathbf{A}_m}\dot{\mathbf{A}}_m - L \\ &= -\varepsilon_0 \left(-\dot{\mathbf{A}}_m - \mathrm{grad}\varphi\right)\dot{\mathbf{A}}_m - \frac{1}{2}\varepsilon_0 \left(-\dot{\mathbf{A}}_m - \mathrm{grad}\varphi\right)^2 + \frac{1}{2\mu_0}\left(\mathrm{curl}\mathbf{A}_m\right)^2 - \sigma \mathrm{curl}\mathbf{\Pi}_m \cdot \mathrm{curl}\mathbf{A}_m \\ &\quad + \frac{1}{2}\sigma^2\mu_0 \left(\mathrm{curl}\mathbf{\Pi}_m\right)^2 + \frac{1}{2}\varepsilon_0 \mathbf{\Pi}_m \mathrm{curlcurl}\nabla^2\mathbf{\Pi}_m + \frac{1}{2}\varepsilon_0^2\mu_0 \left(\mathrm{curl}\dot{\mathbf{\Pi}}_m\right)^2, \end{aligned} \tag{14}$$

Since the Hamiltonian does not depend on time explicitly, thus its volume integral is constant, i.e., the totel energy is conserved during the processes. A simpler expression would be more practical to identify the energy terms. Applying the relations between the measurable physical quantities and the potentials, the Maxwell equations, the Lorenz condition, and taking into account that the Hamiltonian is unambiguous up to a divergence or a time derivative we obtain a compact formula

$$H = \frac{1}{2}\varepsilon_0 \mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2 - \frac{1}{2}\sigma^2\mu_0 \left(\mathrm{curl}\mathbf{\Pi}_m\right)^2. \tag{15}$$

The first two terms give the electric and magnetic field energies. The third term pertains to the dissipative Joule heat loss caused by the conductive current. It seems the generalized Hertz vector has a particular role in the conductive eletromagnetic energy loss process. If an electric conductor does not take place in space than the electromagnetic energy remains constant.

5. SUMMARY

We pointed out that the Hertz vector can have a generalized form by which the Maxwell equations involving the conductive currents can be successfully produced. In this way, the Joule dissipation appears on a potential level. This generalization of the Hertz vector enables us to create the Lagrangian description of such an electromagnetic field in which we can handle the loss of electromagnetic energy. The calculated Hamiltonian of the process clearly shows that the electromagnetic field energy dissipates into Joule heat. If there is no conductive current in the space, the electromagnetic energy is conserved during the process. We hope that based on the presented Lagrangian formulation, the electromagnetic and the thermal fields can couple, by which further studies may be possible in the case of electromagnetic radiation in media. We believe that beyond the mentioned cases in the motivation [2–10], the method can even be extended to materials with magnetic behavior [31].

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7. THE COMPLIANCE WITH ETHICAL STATEMENT

not applicable.

8. CONFLICT OF INTEREST

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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